A GENERAL BOUND FOR OSCILLATORY INTEGRALS WITH A POLYNOMIAL PHASE OF DEGREE \( k \)

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Abstract. Let \( f \in \mathbb{R}[X_1, ..., X_n] \) be a polynomial of degree \( k \geq 2 \). We consider the oscillatory integral \( I(\lambda) = \int \varphi(x) e^{i\lambda f(x)} \, dx \), where \( \varphi \) is a \( C^1 \) function with compact support. A classical result due to E.M. Stein implies that \( I(\lambda) = O(\lambda^{-1/k}) \), as \( \lambda \to +\infty \).

The exponent \( 1/k \) is best possible, as shown by the example \( f(x) = f(x_0) \pm L(x - x_0)^k \), where \( x_0 \) is any point in \( \mathbb{R}^n \) and \( L \) is any nonzero linear form on \( \mathbb{R}^n \). In this paper, we show that, if \( f \) is precisely not of the above form, then the stronger bound \( I(\lambda) = O(\lambda^{-1/(k-1)}) \) holds, and the exponent \(-1/(k - 1)\) is best possible.

1. Statement of the result

We consider a polynomial \( f \in \mathbb{R}[X_1, ..., X_n] \) of total degree \( k \), i.e. \( f(x) = \sum_{\alpha} a_{\alpha} x^\alpha \), with \( \alpha = (\alpha_1, ..., \alpha_n) \), such that \( |\alpha| = \alpha_1 + ... + \alpha_n \leq k \); for each \( \alpha \) in the summation, and such that there exists at least one \( \alpha \) with \( |\alpha| = k \) and \( a_{\alpha} \neq 0 \); here, as usual, we have set \( x^\alpha = x_1^{\alpha_1} ... x_n^{\alpha_n} \). Let us denote by \( C^1_c(\Omega) \) the set of all functions \( \varphi \) which are \( C^1 \) with compact support contained in the open set \( \Omega \). Let \( \Delta \) denote the largest size of those \( |a_{\alpha}| \) for which \( |\alpha| = k \). Then for any \( \varphi \in C^1_c(\mathbb{R}^n) \), the general bound of Stein’s Lemma (cf Lemma 1 below) applies here in the following form:

\[
(1.1) \quad \left| \int \varphi(x) e^{i\lambda f(x)} \, dx \right| \leq C_1(k)(\Delta \lambda)^{-1/k}(\|\varphi\|_{L^\infty} + \|\varphi'\|_{L^1}), \quad \text{for all } \lambda > 0
\]

where \( C_1(k) \) is a positive constant that depends only on the degree \( k \). Such a bound is quite uniform. Our aim is to improve the exponent \(-1/k\) in \(-1/(k - 1)\), providing that \( f \) cannot be written as

\[
(1.2) \quad f(x) = f(x_0) \pm L(x - x_0)^k,
\]

for some \( x_0 \in \mathbb{R}^n \) and some linear form \( L \) on \( \mathbb{R}^n \). If so, we shall say that \( f \) satisfies the hypothesis \( H_{k,n} \). Our analog of (1.1) is as follows.

Theorem 1. We suppose that \( f \) satisfies the hypothesis \( H_{k,n} \) and that \( \Omega \) is a bounded open set in \( \mathbb{R}^n \). We then have, for any \( \varphi \in C^1_c(\Omega) \),

\[
(1.3) \quad \left| \int \varphi(x) e^{i\lambda f(x)} \, dx \right| \leq C_2(f, \Omega) \|\varphi\|_1 \lambda^{-1/(k-1)}, \quad \text{for all } \lambda > 0
\]

where \( C_2(f, \Omega) \) is a positive constant which depends only on \( f \) and \( \Omega \), and where we have set

\[
\|\varphi\|_1 = \|\varphi\|_{L^\infty} + \|\varphi'\|_{L^\infty}.
\]
This bound is far from being as uniform as (1.1), the constant \( C_2(f, \Omega) \) depending on an abstract partition of unity. However, the exponent \(-1/(k-1)\) is best possible, as shown by the example \( f(x, y) = x^{k-1}y \), where in that case the true order of the oscillatory integral (restricted to a neighbourhood of the origin) is given by the stationary phase method, see e.g. Theorem 3 in §8.3, CH.II of [1].

Now, we present a slight generalisation of Theorem 1, which may be of some interest, and which does not increase our proof. We consider a Lebesgue-measurable subset \( \Gamma \) of \( \mathbb{R}^n \) which has the following property:

\[
\text{(1.4)} \quad \text{"For any line } D, \text{ the set } D \cap \Gamma \text{ is the union of at most } N \text{ segments".}
\]

The example we have in mind is the following: \( \Gamma \) is the intersection of a compact convex subset of \( \mathbb{R}^n \) with the set \( \{ x \in \mathbb{R}^n; Q(x) \geq 0 \} \), where \( Q \in \mathbb{R}[X_1, \ldots, X_n] \) has degree at most \( N \).

**Theorem 2.** We suppose that all hypotheses of Theorem 1 are satisfied and that \( \Gamma \) satisfies (1.4). We then have, for any \( \varphi \in C_1^c(\Omega) \):

\[
\left| \int_{\Gamma} \varphi(x)e^{i\lambda g(x)}dx \right| \leq NC_4(\|g\|_{L^\infty}, k)(\|\varphi\|_{L^1} + \|\varphi'\|_{L^1})\lambda^{-1/k} \quad \text{for } \lambda > 0
\]

where \( C_4(f, \Omega) \) is a positive constant which depends at most on \( f \) and \( \Omega \).

Of course, Theorem 2 contains Theorem 1 and the rest of this paper is devoted to its proof.

2. Basic lemmas

We first recall Stein’s fundamental lemma (cf [2], Proposition 5, page 342), with a slight modification concerning the domain \( \Gamma \) of integration.

**Lemma 1.** Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n \) and let \( g : \Omega \to \mathbb{R} \) be a regular function such that the derivative \( \partial^\alpha g = \frac{\partial^k g}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \), with \( k = |\alpha| \), and \( k \geq 1 \), satisfies

\[
|\partial^\alpha g(x)| \geq 1, \quad \text{for all } x \in \Omega
\]

and let \( \Gamma \) satisfy (1.4). Then, for any \( \varphi \in C_1^c(\Omega) \), one has

\[
\left| \int_{\Gamma} \varphi(x)e^{i\lambda g(x)}dx \right| \leq NC_3(\|g\|_{L^\infty}, k, \varphi, \varphi')\lambda^{-1/(k-1)} \quad \text{for } \lambda > 0
\]

where \( C_3(f, \Omega) \) is a positive constant which depends at most on \( f \) and on the maximal size of the derivatives of order \( k + 1 \) of \( g \).

**Proof.** In the case \( N = 1 \), this lemma is exactly Stein’s Lemma. For proving it in the case \( N > 1 \), note that Stein’s proof uses a linear change of variables which does not alter the property (1.4) and which reduces the problem to a one dimensional oscillatory integral, but, in our case, with several (at most \( N \)) intervals. It is thus obvious that the case \( N > 1 \) reduces to the case \( N = 1 \) by multiplying the final bound by \( N \).

The following elementary one dimensional lemma is also needed.
Lemma 2. Let $k \geq 2$ be an integer, $s \geq 1$ be real, $\chi : [a, b] \to \mathbb{C}$ be a $C^1$ function, with $0 \leq a < b \leq 1$. We then have the bound

$$\int_a^b \chi(t)e^{i\lambda t^k}t^s \, dt = O \left( \|\chi\|_{L^1} \lambda^{-1/(k-1)} \right)$$

where the implied constant depends only on $k$.

Proof. We make a change of variable by setting $t = \tau^{(k-1)/k}$, and we get

$$\int_a^b \chi(t)e^{i\lambda t^k}t^s \, dt = \int_{a_1}^{b_1} \chi_1(\tau)e^{i\lambda \tau^{k-1}} \, d\tau,$$

where we have introduced obvious notations. As we have assumed $s \geq 1$ and $k \geq 2$, we have $\|\chi_1\|_{L^1} = O(\|\chi\|_1)$ and $\|\chi_1\|_{L^\infty} = O(\|\chi\|_1)$, so that we may apply the Corollary of Proposition 2 in page 332 of [2].

We need also an immediate algebraic lemma.

Lemma 3. Let $P \in \mathbb{R}[X_1, \ldots, X_n]$, with $n \geq 2$, be a homogeneous polynomial which satisfies

$$P(x_1, \ldots, x_{n-1}, 1) = 0,$$

for all real numbers $x_1, \ldots, x_{n-1}$. Then $P(x) = 0$ for all $x \in \mathbb{R}^n$.

Proof. Of course, applying Taylor’s formula over the $x_n$ variable, we see that $P$ is divisible by $x_n - 1$. □

3. The local form of the theorem

We establish now the main intermediate result in the proof of Theorem 2: we show how to divide the domain of integration according to the local properties of $f$, considering here the worst case.

Theorem 3. Let $P \in \mathbb{R}[X_1, \ldots, X_n]$ be a homogeneous polynomial of degree $k \geq 2$, satisfying the property

$$P(x_1, \ldots, x_{n-1}, 1) = 0, \text{ for all real numbers } x_1, \ldots, x_{n-1}.$$ Then there exists an open neighbourhood $V$ of 0 in $\mathbb{R}^n$ such that, for any set $\Gamma$ satisfying (1.4) and any $\psi \in C^1_c(V)$, one has

$$\left| \int_{\Gamma} \psi(x)e^{i\lambda P(x)} \, dx \right| \leq NC_5(P) \|\psi\|_1 \lambda^{-1/(k-1)},$$

where $C_5(P)$ is a positive constant which depends at most on $P$.

Proof. We divide the proof in several steps.

1) Splitting the domain of integration

We fix a $C^1$ test function $\psi$ whose support is contained in $[-1, 1]^n$. From now on, we shall use the symbol $u \ll v$ to mean that there exists a constant $C$ (which depends at most on $P$ and on other parameters that will be recalled when necessary), such that one has $|u| \leq Cv$. We then set

$$I(\lambda) = \int_{\Gamma} \psi(x)e^{i\lambda P(x)} \, dx$$
and we have to prove that

$$I(\lambda) \ll N \|\psi\|_1 \lambda^{-1/(k-1)},$$

for all $\lambda > 0$, providing that $\psi$ has its support contained in a sufficiently small neighbourhood of 0. We split the domain of integration into $2^n$ parts, writing, for each $\varepsilon = (\varepsilon_1, ..., \varepsilon_n) \in \{-1, 1\}^n$,

$$\Gamma_\varepsilon = \{x \in \Gamma; \varepsilon_j x_j \geq 0, j = 1, ..., n\}$$

so that we have $\Gamma = \bigcup_\varepsilon \Gamma_\varepsilon$. Moreover, for each $\varepsilon$, we split $\Gamma_\varepsilon$ into $n$ parts: for each $r = 1, 2, ..., n$, we define

$$\Gamma_{\varepsilon, r} = \{x \in \Gamma_\varepsilon; |x_j| \leq |x_r| \text{ for } j = 1, ..., n\}$$

so that we have

$$|I(\lambda)| \leq n2^n \max_{\varepsilon, r} \left| \int_{\Gamma_{\varepsilon, r}} \psi(x)e^{i\lambda P(x)}dx \right|.$$

We are going to bound, for instance,

$$I_0(\lambda) = \int_{\Gamma_0} \psi(x)e^{i\lambda P(x)}dx,$$

with $\Gamma_0 = \{x \in \Gamma; 0 \leq x_j \leq x_n, j = 1, 2, ..., n-1\}$

and we have to prove that there exists a neighbourhood $V$ of 0 in $\mathbb{R}^n$ such that the bound

$$I_0(\lambda) \ll N \|\psi\|_1 \lambda^{-1/(k-1)},$$

for all $\lambda > 0$ and for all $\psi \in C^1_c(V)$ holds, the implied constant depending at most on $P$ (and thus on $n$ and $k$).

2) A change of variables

We want to prove Theorem 3 by induction on $n$. We note that, for $n = 1$, there is nothing to prove, because a homogeneous polynomial of degree $k$ in one variable cannot satisfy (3.1). Thus, we suppose $n \geq 2$. If $n \geq 3$, we assume that the theorem have been proved up to the dimension $n-1$, and if $n = 2$, we have nothing to assume, Lemma 1 being a sufficient reference. In order to make a change of variables which will reduce the dimension (in some way, at least), we define the sets $S = \{x \in [0, 1]^n; x_j \leq x_n \text{ for } j = 1, ..., n-1\}$ and $T = [0, 1]^{n-1}$ We set

$$x_1 = tu_1, ..., x_{n-1} = tu_{n-1}, x_n = t, \text{ so that we have the general formula}$$

$$\int_S \chi(x)dx = \int_0^1 \int_T \chi(tu, t)t^{n-1}dtdu.$$
Suppose first that \( Q(a) \neq 0 \), say \( |Q(a)| = 2\delta \), with \( \delta > 0 \). Then we choose \( V(a) \) so small that \( |Q(u)| \geq \delta \) throughout \( V(a) \). For each fixed \( u \in V(a) \), we bound the integral \( \int_0^1 \gamma(tu, t)\psi(tu, t)\exp(i\lambda Q(u)t^k)dt \) by means of Lemma 2; for this, we have to recall that we have fixed \( n \geq 2 \), and to note that the function \( t \to \gamma(tu, t) \) is the characteristic function of a union of at most \( N \) intervals. Integrating then over \( u \), we obtain (3.8) in the case \( Q(a) \neq 0 \).

4) We consider now the more difficult case where \( Q(a) = 0 \). We set \( u = a + v \) and \( R(v) = Q(a + v) \); \( R \) is thus a polynomial in \( n - 1 \) variables, of degree \( \leq k \), with \( R(0) = 0 \).

We write \( R(v) = \sum\alpha a_\alpha v^\alpha \), with \( \alpha = (\alpha_1, ..., \alpha_{n-1}) \in \mathbb{N}^{n-1}, |\alpha| \leq k \). We dismiss the case \( R(v) \equiv 0 \); indeed, this would mean that \( P(x_1 - a_1, ..., x_{n-1} - a_{n-1}, 1) \equiv 0 \), which is impossible by Lemma 3.

Thus, we know that there is at least one index \( \alpha \neq 0 \) such that \( b_\alpha \neq 0 \). We first consider the case where this \( \alpha \) satisfies \( |\alpha| = l \), with \( 1 \leq l \leq k - 1 \).

For each fixed \( t \in [0, 1] \), we note that the function \( u \to \gamma(tu, t) \) is the characteristic function of a set in \( \mathbb{R}^{n-1} \) which satisfies (1.4).

Now, the derivative \( \partial^\alpha R(v) \) is equal to the constant term \( (\alpha_1!)(\alpha_{n-1}!)b_\alpha \) plus non constant monomials that will be small if we restrict \( v \) to a sufficiently small neighbourhood of \( 0 \) in \( \mathbb{R}^{n-1} \). We have shown that there exists a neighbourhood \( W \) of \( 0 \) in \( \mathbb{R}^{n-1} \) and a real \( \delta > 0 \), both depending only on \( a \) and \( P \) (and, in particular, not on \( t \)) such that \( |\partial^\alpha R(v)| \geq \delta \) throughout \( W \).

We set \( V(a) = a + W \), so that we have \( |\partial^\alpha Q(u)| \geq \delta \) throughout \( V(a) \), and we apply Lemma 1:

\[
\int_{T \cap V(a)} \gamma(tu, t)\psi(tu, t)\exp(i\lambda t^k Q(u))du \ll N \|\psi\|_1 (\delta t^k \lambda)^{-1/l},
\]

for all \( \lambda > 0 \) and each \( t \in [0, 1] \).

Integrating this inequality over \( t \), we set \( A(t) = t^{n-1} \min\{1, (t^k \lambda)^{-1/l}\} \), and we write

\[
\int_0^1 A(t)dt = \int_0^\tau A(t)dt + \int_\tau^1 A(t)dt \\
\leq \int_0^\tau t^{n-1}dt + \lambda^{-1/l}\int_\tau^1 t^{n-1-k/l}dt \\
\ll \tau^n (1 + \lambda^{-1/l} \tau^{-k/l}) + \lambda^{-1/l}
\]

In this last bound, we take \( \tau = \lambda^{-1/k} \) (assuming \( \lambda \geq 1 \), otherwise there is nothing to prove), and we get

\[
\int_0^1 A(t)dt \ll \lambda^{-n/k} + \lambda^{-1/l}.
\]

From this, we recover (3.8).

5) For proving (3.8), it remains to consider the case where \( Q(a) = 0 \) and where \( R(v) \) is a homogeneous polynomial of degree \( k \).

But such a situation cannot occur in the case \( n = 2 \). Indeed, \( R(v) \) is a homogeneous polynomial of degree \( k \) and can be written as \( R(v) = bv^k \), and thus, \( P(x, 1) = \)
b(x - a)^k. By Lemma 3, the only polynomial P(x, y), homogeneous of degree k, which satisfies P(x, 1) = b(x - a)^k is P(x, y) = b(x - ay)^k, so that (3.1) is not satisfied.

Now, we suppose n ≥ 3. In the same way as above, we show that R(v) cannot be written as R(v) = ±L(v)^k: otherwise we should have

\[ P(x_1, ..., x_{n-1}, 1) = ±L(x_1 - a_1, ..., x_{n-1} - a_{n-1})^k \]

and this would imply that

\[ P(x_1, ..., x_{n-1}, x_n) = ±L(x_1 - a_1x_n, ..., x_{n-1} - a_{n-1}x_n)^k + P_0(x) \]

where \( P_0(x) \) is a homogeneous polynomial which satisfies \( P_0(x_1, ..., x_{n-1}, 1) = 0 \), for all \( x_1, ..., x_{n-1} \). By Lemma 3, this is possible only if \( P_0(x) \equiv 0 \). Thus \( R \) satisfies (3.1) in the lower dimension \( n - 1 \).

From our recurrence hypothesis (that the theorem is true in the \( n - 1 \) dimensional case), there exists a neighbourhood \( W \) of 0 in \( \mathbb{R}^{n-1} \) such that, setting \( V(a) = a + W \), we have

\[
\int_{T \cap V(a)} \gamma(tu, t)\psi(tu, t)e^{i\lambda^kQ(a)}du \ll N\|\psi\|_1 (t^k\lambda)^{-1/(k-1)}. \tag{3.10}
\]

We integrate this inequality over \( t \) as previously (see the corresponding proof in step 4), and we recover (3.8).

We have finally proved (3.8) unconditionally when \( n = 2 \), and also for \( n > 2 \), providing that the theorem is true in dimension \( n - 1 \).

6) Conclusion

We treat together the cases \( n = 2 \) and \( n > 2 \) because they are identical, but one should have to prove firstly the case \( n = 2 \), and then, the case \( n > 2 \) by induction on \( n \).

We have to prove (3.5). For each \( a \in T \), we choose a neighbourhood \( V(a) \) as in (3.8). Let \( \chi_1, ..., \chi_s \) be \( C^1 \) functions on \( \mathbb{R}^{n-1} \), each one having his support included in one of the \( V(a) \), and such that \( \sum_{r=1}^{s} \chi_r(u) = 1 \) for all \( u \in T \). We have

\[ I_0(\lambda) = \sum_{r=1}^{s} \int_{T} \gamma(tu, t)\chi_r(u)\psi(tu, t)e^{i\lambda^kQ(a)}t^{n-1}dtdu. \]

We bound each integral in the sum using (3.8) and we obtain (3.5). The proof is complete.

4. Proof of Theorem 2

Let \( f, \varphi, \Omega \) and \( \Gamma \) be as in Theorem 2. For each \( x_0 \) in the compact \( \Omega \), we are going to construct a neighbourhood \( V(x_0) \) of \( x_0 \) in \( \mathbb{R}^n \), so that, for each \( \chi \in C_0^1(V(x_0)) \) and each \( \lambda > 0 \), we have

\[
\int_{\Gamma} \chi(x)e^{i\lambda f(x)}dx \ll N\|\chi\|_1 \lambda^{-1/(k-1)}, \tag{4.1}
\]

where the implied constant depends at most on \( f \). Assuming (4.1), it is easy to deduce (1.5) with a partition of unity, in the same way as above. Now, our aim is to prove (4.1).
We fix $x_0 \in \Omega$. We set $P(y) = f(y + x_0) - f(x_0)$; $P$ is a polynomial of degree $k$, which is not of the form $\pm L(y)^k$. We have to find a neighbourhood $W$ of 0 in $\mathbb{R}^n$ such that we have

$$\int_{\tilde{\Gamma}} \chi(y)e^{\alpha \lambda P(y)}dy \ll N \|\chi\|_1 \lambda^{-1/(k-1)},$$

for $\chi \in C^1_c(W)$ and $\lambda > 0$,

where we have set $\tilde{\Gamma} = -x_0 + \Gamma$, and where the implied constant depends at most on $f$.

As $P$ vanishes at 0, either $P$ is a homogeneous polynomial of degree $k$, or we have

$$P(y) = \sum_{\alpha} \alpha \cdot y^\alpha,$$

where $a_\alpha \neq 0$ for some $\alpha$ with $1 \leq |\alpha| \leq k - 1$.

In the first case, which is the more difficult, (4.2) is precisely the conclusion of Theorem 3, so that we may suppose that (4.3) holds. Let $\alpha \in \mathbb{N}^n$ such that $1 \leq |\alpha| \leq k - 1$, and $|\alpha_\alpha| = 2\delta$ for some $\delta > 0$. Then there exists a sufficiently small neighbourhood $W$ of 0 in $\mathbb{R}^n$ such that the derivative $\partial^\alpha P$ satisfies $|\partial^\alpha P(y)| \geq \delta$ throughout $W$.

Thus we may apply Lemma 1 and this implies precisely (4.2). The proof of Theorem 2 is now complete.

References
