Zeta functions and solutions of Falconer-type problems for self-similar subsets of \( \mathbb{Z}^n \)

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Abstract

This paper uses the zeta function methods to solve Falconer-type problems about sets of \( k \)-simplices whose endpoints belong simultaneously to a self-similar subset \( \mathcal{F} \) of \( \mathbb{Z}^n \) \((k \leq n)\) and a disc \( B(x) \) of a large radius \( x \). Assuming that the similarity transformations pairwise commute, we study four Euclidean metric invariants of the simplices, the most basic (and frequently studied) of which is the distance between endpoints of a 1-simplex. For each, we introduce a zeta function, derive its functional properties, and apply such information to derive a lower bound on the upper Minkowski dimension of \( \mathcal{F} \), which guarantees that the number of distinct metric invariants must be unbounded as \( x \to \infty \).

1. Introduction

This article uses zeta function methods to study the discrete analog of Falconer-type problems for unbounded self-similar sets \( \mathcal{F} \subset \mathbb{Z}^n \). Given \( \mathcal{F} \), a Euclidean metric invariant defined on the set of \( k \)-simplices \((1 \leq k \leq n)\), and a disc \( B(x) \) of a large radius \( x \), a (discrete) Falconer-type problem asks for a threshold on the upper Minkowski dimension of \( \mathcal{F} \), above which the number of distinct invariant values, determined by simplices whose endpoints belong to \( \mathcal{F} \cap B(x) \), must be an unbounded function of \( x \) when \( x \to \infty \). As such, a solution, for our purposes, is always asymptotic in nature.

Our basic observation is that zeta function methods are very well suited to solve such problems when \( \mathcal{F} \) is ‘compatible’, that is, a self-similar set whose similarities pairwise commute. For each of the four invariants defined below, we prove that the number of distinct invariants must be, at least, an explicit positive power of \(#(\mathcal{F} \cap B(x))\) if the upper Minkowski dimension of \( \mathcal{F} \) exceeds a simple lower bound that depends only upon \( n \).

We are able to prove our theorems by combining the analysis of an underlying zeta function with diophantine methods over \( \mathbb{Z} \) that estimate the density of integral points on classes of fairly simple algebraic sets. Indeed, to find each of the lower bounds (for the upper Minkowski dimension) needed to solve the Falconer problem, it suffices to impose the condition that a certain difference between a quantity analytic in nature and a quantity diophantine in nature must be positive.

We first choose a vector \( \mathbf{m} = (m_1, \ldots, m_{k+1}) \in \mathcal{F}^{k+1} \) \((\mathbf{m}_i \neq 0 \ \forall i)\) and define the \( k \)-simplex

\[
\Sigma_k(\mathbf{m}) = \langle m_1 - m_{k+1}, \ldots, m_k - m_{k+1} \rangle := \left\{ \sum_{\ell \leq k} \eta_\ell (m_\ell - m_{k+1}) : \sum_\ell \eta_\ell = 1 \text{ and } \eta_\ell \geq 0 \ \forall \ell \right\}.
\]

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Four different metric invariants of $\Sigma_k(m)$ interest us.

1. ($k = 1$) Distance between two points.
2. ($k = 2$) The angle of a 2-simplex determined by the two unit vectors $m_1/\|m_1\|$ and $m_2/\|m_2\|$.
3. ($k \geq 2$) The configuration rooted at $m_{k+1}$ for any $k \geq 2$, that is, the vector of individual lengths $(\|m_1 - m_{k+1}\|, \ldots, \|m_k - m_{k+1}\|)$, where each distance is assumed to be positive.
4. ($k = n$) The volume of $\Sigma_n(m)$.

We have chosen these four invariants because they offer a good illustration of how zeta functions can be used to address Falconer-type problems, a point that does not yet seem to have been observed in the literature. Other invariants can also be analyzed via this technique. Because these four invariants have also been studied by other techniques, their study also allows us to compare the relative utility of our method.

The statements of all the results we prove in this article are given in § 2.1. It is important to realize that all our results are unconditional and apply for any $n \geq 2$, except for that involving angles which requires $n$ to be at least 4. This is in distinct contrast to the conditional results we described in [6, 7], which assumed that results of Mattila could extend to unbounded discrete self-similar sets. By restricting our attention to subsets of $\mathbb{Z}^n$, and using diophantine arguments that estimate the number of integral points on some simple algebraic sets of small degree, we avoid having to impose any supplementary hypothesis from geometric measure theory.

An overview of the results follows a description of the context in which we think it is helpful to place them.

The original context for Falconer-type problems is the geometric measure theory of bounded sets $E \subset \mathbb{R}^n$ with positive Hausdorff measure. The inspiration of a large body of work has been to prove a well-known conjecture of Falconer, which predicts that a certain lower bound on the Hausdorff dimension of $E$ guarantees that the distance set

$$\Delta(E) = \{\|x_1 - x_2\| : x_1, x_2 \in E\}$$

has positive Lebesgue measure.

For dimensions $n > 2$, there is, however, no particularly compelling reason to be complacent and limit oneself to lengths of line segments. In particular, it would seemingly be more natural to work with simplices and replace distance by volume or configuration sets. The articles [9–11, 17] have all been motivated to do just this by formulating and proving variants of Falconer-type problems, for example, for

$$\text{Vol}(E) = \{\text{vol}(\Sigma_n(x)) : x = (x_1, \ldots, x_n) \in E^n\}.$$ 

The question then becomes how large must the Hausdorff dimension of $E$ be to insure that the Lebesgue measure of Vol(E) is positive.

One other goal of such work has been to answer the same discrete Falconer-type problem that is addressed in this paper, albeit for different classes of unbounded discrete sets $E$ (neither self-similar nor contained in $\mathbb{Z}^n$). In order to apply any result about bounded sets to discrete unbounded sets $E$, the idea common to all these papers is to reduce the problem to families of sufficiently small neighborhoods of bounded discrete sets by a natural scaling/normalization procedure applied to a subset $E_k$ of cardinality $k$. This offers the opportunity to apply a large arsenal of powerful tools from both harmonic analysis and geometric measure theory to the uniformly bounded neighborhoods. One can then try and bootstrap one’s way back to say something about the number of distinct metric invariants determined by the points in $E_k$ as $k \to \infty$.

This procedure has been useful, but in order for it to work one also needs to impose an additional hypothesis of a geometric measure theoretic nature. In particular, for each of the
applications to discrete sets in the papers cited above, it is necessary to impose the condition called ‘s-adaptability’ (for s larger than the Hausdorff dimension of the neighborhoods). Such a property, while plausible or technically appealing, does not seem all that easy to verify starting with unbounded self-similar sets of the type we study.

One consequence of our work is that it seems well suited to address discrete analogs of Falconer-type problems without having to impose a priori the s-adaptability hypothesis. For our purposes here, it suffices for the set to be compatible, a property very easy to verify, in order to exploit the zeta function method.

Furthermore, we believe that comparable results can also be proved for self-similar subsets of \( \mathbb{Z}^n \) that are not compatible since we do not yet see a fundamental obstacle to extending the methods in [6] to non-compatible self-similar sets. Indeed, even for such a set, there is a meromorphic continuation of its zeta function, as well as a generalized Dirichlet polynomial (see §2.2.1) whose roots include the poles of the zeta function belonging to some neighborhood of the boundary of absolute convergence. What still remains to be done is to prove Lemma 1 (§2.2.1) for this set of roots (see the Remark at the end of §2.2.2 for some additional details). We hope to be able to say more about this in the near future.

Our first result addresses the distinct distance problem of Erdős in an asymptotic sense and for any \( n \geq 2 \). This seems to us a good first test case to see what the zeta function method is capable of. When \( \mathcal{F} \subset \mathbb{Z}^2 \) is compatible, the exponent we derive for the lower bound of the number of distinct distances between points in \( \mathcal{F} \cap B(x) \) for all large \( x \) is almost as good as that predicted by Erdős, that is, \( 1 - \varepsilon \) for any \( \varepsilon > 0 \). However, we do not see any way by which our method could do any better than this and give the actual exponent appearing in Erdős’ original paper.

It is perhaps also worthwhile to point out here, since there may be some confusion on this point, that a recent article of Orponen [21] shows that the Hausdorff measure of the distance set determined by a self-similar subset of \( \mathbb{R}^2 \) equals 1. However, this paper does not show that the Lebesgue measure of the distance set is positive, a property that is needed to exploit the ‘conversion method’ of Iosevich–Laba [16], which shows how a solution of the Falconer conjecture implies a solution to the Erdős distance problem.

When \( n > 2 \), we show that the zeta function method is capable of improving upon the asymptotic result [22] when the upper Minkowski dimension of \( \mathcal{F} \) belongs to the interval \( (n \cdot (n^2 - 4)/(n^2 - 1), n] \). On the other hand, if the upper Minkowski dimension is strictly less than \( n \cdot (n^2 - 4)/(n^2 - 1) \), then our exponent of growth is weaker than that in [22]. These statements are given in Theorem 1 and proved in §3.1. What is of most interest in Theorem 1 is that our result finds a small interval for the upper dimension in which the exponent we derive for the asymptotic problem is actually superior to that given in [22].

A natural variant of considering the length of a 1-simplex determined by \( \mathbf{m}_1, \mathbf{m}_2 \in \mathcal{F} \cap B(x) \) is the angle of the 2-simplex formed by connecting the origin to the two unit vectors in the directions of \( \mathbf{m}_1, \mathbf{m}_2 \). Assuming \( n \geq 4 \), we derive a simple lower bound on the upper Minkowski dimension, which insures that the number of distinct cosine values must grow, at least, to some positive power of \#\( \mathcal{F} \cap B(x) \).

As was noted in [17], little is known, in general, about the distribution of angles determined by points in unbounded discrete subsets of \( \mathbb{R}^n \) once \( n \geq 4 \). Whereas the results proved in [17] require the hypothesis of s-adaptability, we give a solution to a Falconer problem for angles for any compatible self-similar subset of \( \mathbb{Z}^n \) once \( n \geq 4 \) with no other hypothesis. The statement is given in Theorem 2 and proved in §3.2. It is perhaps of interest to note here that the diophantine component of our proof only uses an elementary result about lattice points on absolutely irreducible quadratic hypersurfaces.

The metric invariants in (3), (4) are examples of how one might hope to extend the Erdős distance problem to study the metric geometry of configurations or simplices determined by more than two points in higher-dimensional contexts. Theorems 3 and Corollary 1 present our
results concerning distinct numbers of configurations. Proofs are in §3.3. Here the technique is similar to that used in §3.1. Sections 4 and 5 study the distinct volume problem (4). The statement of the result is given in Theorem 4 and the proof is given in §5.

The key ingredient for all our results is a generalization of a classic Tauberian procedure for one variable Dirichlet series with positive coefficients

\[ f(s) = \sum_{n} \frac{a_n}{\lambda_n^s} \] (where \(0 < \lambda_1 < \lambda_2 < \cdots\) is an unbounded sequence)

to Dirichlet series in two or more variables whose coefficients are the (squares of) values of the invariants (1)–(4) that interest us. For example, for invariant (1) this leads to a two-variable series

\[ \zeta_2(s_1, s_2) := \sum_{(m_1, m_2) \in (F-\{0\})^2} \frac{\|m_1 - m_2\|^2}{\|m_1\|^s_1\|m_2\|^s_2}. \] (1.1)

In the classic case, the weakest effective description of a dominant asymptotic behavior (for all large \(x\)) of the average is given by

\[ A(x) := \sum_{\lambda_n < x} a_n = Cx^\alpha \ln^\ell x + o(x^\alpha \ln^\ell x), \] where \(C \neq 0\). (1.2)

This follows once one knows that \(f(s)\) is absolutely convergent in a half plane, has a meromorphic continuation to some larger region, and has a single real pole on the boundary of analyticity. The value of \(\alpha\) equals this pole.

The idea is to say something equally precise about both the meromorphic behavior of the several variable series outside a domain of absolute convergence and the asymptotic behavior of the averages of their coefficients.

The first part requires us to identify precisely a certain finite set of extremal points on the boundary of the domain of analyticity. The set of such points is the natural analog of the largest real pole for a one variable Dirichlet series. This is because they all belong to the vertex set of a polygon (or more generally a polyhedron) that equals the real part of the boundary of analyticity of the pertinent Dirichlet series. We then refer to such points as belonging to the ‘initial polar locus’ of the series.

The second part is Tauberian, and aims to extend (1.2) in a natural way in order to describe a precise asymptotic lower bound for the averages of the coefficients of the several variable Dirichlet series. A lower bound (with positive exponent for \(x\)) suffices since the solution to a Falconer-type problem only requires a non-trivial asymptotic lower bound for the number of distinct invariant values.

The multivariate Tauberian theorem we prove in Theorem 8 (see §2.3) is both new and essential to our method. We are unable to find anything else in the literature that appears even remotely capable of proving the fundamental lower bound estimate (2.45), which is used to prove all the six results stated in §2.1. One of the reasons for this gap in the literature is that the polar locus of our Dirichlet series is, in general, rather more complicated than what has traditionally been studied. This has required us to fill in and verify many technical details (in §2) that cannot be found elsewhere.

An intriguing feature of this approach, it seems to us, is the fact that this classical-based method, when combined with the diophantine geometric upper estimates for lattice points on spheres, can almost prove, in an asymptotic sense, the lower bound predicted by the Erdös conjecture for compatible self-similar sets inside \(\mathbb{Z}^2\). This is due to the fact that the lower bound on the upper Minkowski dimension of \(F\) only needs to be larger than \(n - 2\) for our result to follow. When combined with our, admittedly small, improvement in [22], this suggests that the method can be helpful to address similar problems for sets in Euclidean spaces of any dimension, and for metric invariants (as well as other Riemannian metrics) that are definable.
by other polynomial (or analytic) expressions. Our work on angles, configurations, and volumes is evidence of its potential applicability.

To appreciate further the content here, it is useful to emphasize an important difference, from our perspective, between the problems involving distance and volume. The source of the difference is that the appropriate zeta function to use for volumes, which we call a simplex zeta function (see §4), has an initial polar locus that is rather more difficult, in general, to describe precisely.

This is critical because without a sufficiently precise description, it is usually not possible to use zeta function methods to prove explicit (and non-trivial) lower bound asymptotic estimates of the type needed to solve a Falconer-type problem.

This obstacle obliges us to introduce a property, which we call ‘thickness’. A self-similar set that is thick has a simplex zeta function for which the needed precision (in the initial polar locus description) is available. This is the point of the discussion in §4.2 (see Theorem 9).

What is not, however, evident is how common a property is thickness, or if it even occurs at all. The purpose of §4.3 is to exhibit three very concrete examples of discrete self-similar sets that are thick. It merits remarking here that any Pascal triangle mod $p$ is a thick compatible self-similar set. This property, curiously enough, does not seem to have been thought of before in the literature devoted to this much-studied fractal-like object. Theorem 5 gives an asymptotic result about this family of fractal like sets that is new. Our final result, Theorem 6 extends this result to all Pascal pyramids mod $p$ where multinomial coefficients mod $p$ replace binomial coefficients, and where $p$ is sufficiently large relative to $n$.

Section 5 then applies the preceding discussion to the metric invariant of volumes in (4). The question, to which we provide at least one reasonably general answer, is the following Falconer-type problem:

*Given a discrete self-similar set $F$, how small can the upper Minkowski dimension of $F$ be so that the number of distinct volumes of $n$-simplices, whose endpoints belong to $F \cap B(x)$, increases without bound as $x \to \infty$.*

When $F \subset \mathbb{Z}^n$ is both thick and compatible, we answer this question in §5 (Theorem 4). It suffices for the upper Minkowski dimension to be larger than $n - 1$. This is, in a general sense, the most complete result possible because any $n$-simplex of a self-similar set that lies inside a hyperplane has exactly the same volume, namely 0. Such a self-similar set can only have upper Minkowski dimension at most $n - 1$.

This result can be thought of as a discrete version of [11, Theorem 1.6] that is proved for bounded Salem subsets of $\mathbb{R}^n$. One possible way to think of this comparison is that the Salem property does for Fourier-based methods what thickness does for zeta function methods.

Unlike the arguments of §3, where the diophantine component of our method relies upon uniform, and very good, upper bound estimates for the number of lattice points on spheres of any radius, this component of our method exploits a comparable uniform but essentially elementary bound for the number of lattice points on hyperplanes. This is the basis for our lower bound condition on the upper Minkowski dimension in Theorem 4.

Denoting the upper Minkowski dimension of $F$ by $e_F$, the actual exponent in the lower bound of distinct volumes is, however, weaker. The value of $1 - (n - 1)/e_F - \varepsilon$ is rather far from $1/e_F - \varepsilon$ for any $\varepsilon > 0$. This value when $e_F$ is close to $n$. It does not, however, seem implausible to believe that it could be improved by finding good estimates for the density of lattice points on the determinantal varieties that actually appear in the diophantine problem. Extending such estimates to the highly singular algebraic sets encountered with the volume invariant could improve some of our results. It does not seem too unreasonable to believe that the pioneering work of [3] can be pushed further to do just that.
The analogy between Salem sets and thickness clearly deserves to be understood more completely. It also justifies trying to understand how frequently the latter property is actually encountered when working with self-similar sets. In an article currently in preparation, we explore this question, and show that thickness is an ‘asymptotically generic’ property when confined to classes of compatible self-similar sets that Strichartz has called non-overlapping (see [23]). A useful problem for further work, it seems to us, would be to find more intrinsic characterizations of the thickness property, connecting it to underlying symmetries as done in §4.3 for the permutation group. This should also help identify additional examples of thick self-similar subsets of some $\mathbb{Z}^n$ that occur naturally in combinatorial geometry or number theory. Our proof of the asymptotic genericity property suggests us that there should be many such examples.

2. Statements of quantitative results, technical preliminaries, and a multivariate Tauberian theorem

2.1. Basic definitions and statements of results

We first recall the basic notion of a compatible self-similar subset of $\mathbb{R}^n$ from [6].

We fix in the sequel $(E,q)$ a Euclidean space, $\dim_{\mathbb{R}} E = n$, with standard Euclidean norm $\| \cdot \| = q^{1/2}$, and bilinear form $B(x,y) = \langle x, y \rangle$ the usual scalar product.

**Definition 1.** Let $T_i$ ($i \in I$) be a set of orthogonal linear transformations of $(E,q)$ that pairwise commute. A family $f_i = c_i T_i + b_i$ ($i \in I$) of similarities of $E$ is then said to be ‘compatible’. The constants $c_i$ are the ‘scale factors’ of the similarities. There then exists a uniformizing basis $B = \{ g_j \}$ of $\mathbb{E}$ such that for each $i \in I$ and $j$, there exists $\lambda_{i,j} \in S^1$ so that $T_i^* (g_j) = \lambda_{i,j} g_j$. We then define, for each $i \in I$, the vector $\lambda_i = (\lambda_{i,1}, \ldots, \lambda_{i,n})$.

**Definition 2.** Let $F$ be a countable discrete subset of $E$. Define the upper Minkowski dimension of $F$ by

$$u \dim_{M} F := \lim_{R \to \infty} \frac{\ln |\# F(R)|}{\ln R} \in [0, \infty], \quad \text{where } F(R) := F \cap B(R) \ \forall R.$$ 

We say that $F$ has finite upper Minkowski dimension whenever the limit is finite.

**Remark.** In [6, 7], we used the term ‘exponent’ but in retrospect, this seems to have been unnecessarily confusing.

In this case, the zeta function of $F$ is a series summed over $F' := F - \{ 0 \}$

$$\zeta(F; s) := \sum_{m \in F'} \| m \|^{-s} \quad (2.1)$$

that converges absolutely in the half plane $\sigma := \Re s > u \dim_{M} F$, and $s = u \dim_{M} F$ is its abscissa of convergence.

**Definition 3.** A countably infinite discrete subset $F \subset E$ is said to be a compatible self-similar set if these two properties are satisfied:

(i) $0 < u \dim_{M} F < \infty$;

(ii) there exists a finite compatible set $f = \{ f_i \}_{i=1}^r$ of affine similarities such that each scale factor $c_i > 1$ and where the notation $F \equiv G$ means that $(F \setminus G) \cup (G \setminus F)$ is a finite set,

$$F \equiv \bigcup_{i=1}^r f_i(F) \quad \text{and} \quad f_i(F) \cap f_{i'}(F) \text{ is finite if } i \neq i'.$$  \quad (2.2)
For convenience, and to be consistent with the notation used throughout [6, 7], we will set
\[ e_{\mathcal{F}} := u \dim_M \mathcal{F}. \]

There is a simple analytic characterization of \( e_{\mathcal{F}} \) (see [6, Theorem 2]) as follows:
\[ e_{\mathcal{F}} \] is the unique real solution of the equation
\[ \sum_{j=1}^{r} c_{j}^{-s} = 1. \tag{2.3} \]

In the rest of this article, we use the notation \( \mathcal{F} \) to denote a compatible discrete self-similar subset of \( \mathbb{R}^n \). We also set
\[ D_{\mathcal{F}} := e_{\mathcal{F}} + 2. \tag{2.4} \]

**Statements of our quantitative theorems.**

We prove six explicit lower bound results, each of an asymptotic nature in a large parameter \( x \), for the metric invariants defined in § 1. Despite the variety of results, each can be stated very concisely.

For any \( x > 0 \), we first define (see Definition 2)
\[ \text{Dis}_{\mathcal{F}}(x) := \# \{ \| m_1 - m_2 \| : m_1, m_2 \in \mathcal{F}(x) \}, \tag{2.5} \]
with the understanding that \( \text{Dis}_{\mathcal{F}}(x) \) counts each distance exactly once.

We prove in § 3.1 the following.

**Theorem 1.** Assume \( \mathcal{F} \) is a compatible self-similar set inside \( \mathbb{Z}^n \) \( (n \geq 2) \) with upper Minkowski dimension \( e_{\mathcal{F}} > n - 2 \). Then for any sufficiently small \( \varepsilon > 0 \),
\[ \text{Dis}_{\mathcal{F}}(x) \gg \varepsilon \left[ \# \mathcal{F}(x) \right]^{1-(n-2)/e_{\mathcal{F}}-\varepsilon} \quad \text{as} \quad x \rightarrow \infty. \tag{2.6} \]

**Note:** For simplicity, we will simply write ‘for sufficiently small \( \varepsilon \)’ to mean that \( \varepsilon \) should be smaller than a certain linear expression involving \( e_{\mathcal{F}} \) and \( n \) (that depends upon the particular problem under consideration) in order that the exponent of a monomial in \( x \) is positive.

It is also natural to want an extension of Theorem 1 to angles formed by pairs of points with a fixed origin. To this end, we define for \( m_1, m_2 \in \mathcal{F} \):
\[ \theta(m_1, m_2) = \text{the angle formed between } \frac{m_1}{\| m_1 \|} \text{ and } \frac{m_2}{\| m_2 \|}. \tag{2.7} \]
where we think of the two unit vectors as belonging to the unit sphere with center the origin \( 0 \).

We then define
\[ \text{Ang}(x) = \# \{ \cos(\theta(m_1, m_2)) : m_1, m_2 \in \mathcal{F}(x) \}. \tag{2.8} \]
We prove in § 3.2 the following.

**Theorem 2.** Assume \( \mathcal{F} \) is a compatible self-similar set inside \( \mathbb{Z}^n \) with upper Minkowski dimension \( e_{\mathcal{F}} > n - 2 \). If \( n \geq 4 \), then for any sufficiently small \( \varepsilon > 0 \):
\[ \text{Ang}(x) \gg \varepsilon \left[ \# \mathcal{F}(x) \right]^{1-(n-2)/e_{\mathcal{F}}-\varepsilon} \quad (x \gg 1). \tag{2.9} \]

We prove in § 3.3 a natural extension of Theorem 1 to rooted configurations. Define
\[ \text{Root}_{k, \mathcal{F}}(x) := \# \{ (\| m_1 - m_{k+1} \|, \ldots, \| m_k - m_{k+1} \|) : m_j \in \mathcal{F}(x) \forall j \text{ and } m_j \neq m_{k+1} \forall j \leq k \}. \tag{2.10} \]
THEOREM 3. Assume $\mathcal{F}$ is a compatible self-similar inside $\mathbb{Z}^n$ ($n \geq 2$) with upper Minkowski dimension $e_{\mathcal{F}} > n - 2$. Then for sufficiently small $\varepsilon$, \[
abla_{k,\mathcal{F}}(x) \gg_{\varepsilon} \left|\#\mathcal{F}(x)\right|^{k[1-(n-2)/e_{\mathcal{F}}]-\varepsilon} \quad \text{as } x \to \infty. \tag{2.11}\]

An immediate consequence of Theorem 3 is an extension to arbitrary $k$-point configurations. Define
\[
\text{Config}_{k,\mathcal{F}}(x) := \#\{\left(\|m_1 - m_{k+1}\|, \ldots, \|m_k - m_{k+1}\|, \|m_1 - m_k\|, \ldots, \|m_{k-1} - m_k\|, \ldots, \|m_1 - m_2\|\) : m_j \in \mathcal{F}(x) \forall j \text{ and } m_i \neq m_j \forall i < j\}. \tag{2.12}\]

COROLLARY 1. Assume $\mathcal{F}$ is a compatible self-similar inside $\mathbb{Z}^n$ ($n \geq 2$) with upper Minkowski dimension $e_{\mathcal{F}} > n - 2$. Then for sufficiently small $\varepsilon$, \[
\text{Config}_{k,\mathcal{F}}(x) \gg_{\varepsilon} \left|\#\mathcal{F}(x)\right|^{k(k+1)/2[1-(n-2)/e_{\mathcal{F}}]-\varepsilon} \quad \text{as } x \to \infty. \tag{2.13}\]

To state the last three results, we first define for any vector $m = (m_1, \ldots, m_{n+1}) \in \mathcal{F}^{n+1}$ and $n$-simplex $\Sigma_n(m)$ (see §1), \[
|\Sigma_n(m)| = |\det(m_1 - m_{n+1}, \ldots, m_n - m_{n+1})|, \tag{2.14}\]
and set \[
\text{Vol}_{n,\mathcal{F}}(x) := \#\{\left|\Sigma_n(m)\right| : m \in \mathcal{F}(x)^{n+1}\} \tag{2.15}\]

We define in §4.1 what we mean by thickness of a self-similar set (see Definition 4), and prove in §5 the following three results.

THEOREM 4. Assume that $\mathcal{F} \subset \mathbb{Z}^n$ is thick and compatible, and $e_{\mathcal{F}} > n - 1$. Then for sufficiently small $\varepsilon > 0$, \[
\text{Vol}_{n,\mathcal{F}}(x) \gg_{\varepsilon} \left|\#\mathcal{F}(x)\right|^{1-(n-1)/e_{\mathcal{F}}-\varepsilon} \quad \text{as } x \to +\infty. \tag{2.16}\]

In §4.3, we show that Pascal’s triangle mod any prime $p$, denoted $\text{Pas}(p)$, is a thick subset of $\mathbb{Z}^2$. It is also known that its upper Minkowski dimension $e_p$ is given by \[
e p = \ln(p(p+1)/2)/\ln p. \]

Since $e_p > 1$, Theorem 4 applies since it follows directly from the construction of the set that $\text{Pas}(p)$ is compatible. So, we conclude as follows.

THEOREM 5. For any prime $p$ and sufficiently small $\varepsilon$, \[
\text{Vol}_{2,\text{Pas}(p)}(x) \gg_{\varepsilon} \left|\#\text{Pas}(p)(x)\right|^{1-1/e_p-\varepsilon} \quad \text{as } x \to \infty. \tag{2.17}\]

Since $e_p \to 2$ when $p \to \infty$, we also conclude that (2.17) implies \[
\text{Vol}_{2,\text{Pas}(p)}(x) \gg_{\varepsilon} \left|\#\text{Pas}(p)(x)\right|^{1/2-\varepsilon} \quad \text{as } x \to \infty \quad \text{and for all } p \gg_{\varepsilon} 1. \tag{2.18}\]

It appears that both of these lower bounds describe new features about the fractal-like sets $\text{Pas}(p)$. Moreover, it is perhaps surprising to discover that there is a simple monomial lower
bound for the number of distinct areas of triangles, as counted by $\text{Vol}_2 Pas(p)(x)$, whose exponent is uniform over all sufficiently large $p$ in the sense that for any $\varepsilon > 0$ (and sufficiently small), we can choose this exponent to equal $\frac{1}{2} - \varepsilon$ whenever $p \geq p_\varepsilon$.

In § 4.3, we also show that the Pascal pyramid mod $p$, denoted $M_{n,p}$, is thick if $p \gg n$. These sets are natural generalizations of $Pas(p)$ that use multinomial coefficients instead of binomial coefficients. The upper Minkowski dimension of these sets is denoted by $e_{n,p}$. We showed in [6] that

$$e_{n,p} = \frac{\ln \left( \frac{p + n - 1}{p - 1} \right)}{\ln p}.$$ 

**Theorem 6.** If

$$\left( \frac{p + n - 1}{p - 1} \right) > p^{n-1},$$

then for any sufficiently small $\varepsilon > 0$:

$$\text{Vol}_{M_{n,p}}(x) \gg \varepsilon [\#M_{n,p}(x)]^{1 - (n-1)/e_{n,p} - \varepsilon} \text{ as } x \to +\infty.$$ (2.18)

**Remark.** It easy to see that

1. $n = 2$ implies (2.17) holds for any $p$;
2. $n = 3$ implies (2.17) holds for any $p \geq 3$;
3. $n \geq 4$ implies (2.17) holds for any $p \geq n!$.

2.2. Two technical preliminaries

2.2.1. Properties of Dirichlet polynomials with complex coefficients. Let $c = (c_1, \ldots, c_r) \in (1, \infty)^r$ and $\gamma = (\gamma_1, \ldots, \gamma_r) \in \mathbb{C}^r$. We define a Dirichlet polynomial with complex coefficients

$$\Lambda(c, \gamma, s) = \sum_{j=1}^r \gamma_j c_j^{-s}. $$

We first recall from (2.3) that $e_F$ is the unique real solution of the equation $\sum_{j=1}^r c_j^{e_F} = 1$. It follows that the line $\{\sigma = e_F\}$ is the rightmost vertical line that can first contain a root of $\Lambda(c, 1, s)$, in the sense that there are no roots in the half plane $\sigma > e_F$.

The following Lemma 1 gives us useful information about the set of roots of $\Lambda(c, \gamma, s)$ lying in some neighborhood of the vertical line $\{\sigma = e_F\}$ when $\gamma \in (S^1)^r$. This will be used in § 2.2.2.

Parts (i)–(iii) of the Lemma 1 were proved in [20, Chapter 3] for $\gamma \in (0, \infty)^r$. The proofs extend without significant difficulty to vectors $\gamma \in (S^1)^r$. Parts (iv), (v) of Lemma 1 extend the discussion in [20] (see Theorems 3.25 and 3.26) since this only seems to apply to $\gamma \in \mathbb{N}^r$.

The reader may also find it helpful to consult the article [8] that adds additional and useful details to those given in [20].

As a result, the proofs of Parts (iv), (v) in Lemma 1 in the case $\gamma \in (S^1)^r$ appear to be both new and needed. The essential idea is that we can reduce this case to that of $\gamma = 1$ by a translation of $s$ (along the imaginary axis).

We first recall from [19, 20] that there are two distinct cases ‘latticelike case’ and ‘non-latticelike case’. This depends solely upon the behavior of the roots of the polynomial $\Lambda(c, 1, s)$.

The latticelike case corresponds to the case where the subgroup of $\sum_{j=1}^r \mathbb{Z} \ln c_j$ of $\mathbb{R}$ generated by $\ln c_1, \ldots, \ln c_r$ is discrete. The non-latticelike case corresponds to the possibility where this group is dense in $\mathbb{R}$.
Lemma 1. Assume that $\gamma \in (S^1)^\tau$. The following properties are then satisfied by the roots of $\Lambda(c, \gamma, s)$.

(i) Each root lies in the half plane $\{\sigma \leq e_{F}\}$.
(ii) Any root on the line $\{\sigma = e_{F}\}$ is a simple zero.
(iii) Defining the number of zeroes up to height $T$

$$N_{c, \gamma}(T) = \{ s \in \mathbb{C}; \; \Lambda(c; \gamma; s) = 0 \text{ and } |s| \leq T \}$$

we have the estimate

$$N_{c, \gamma}(T) = O(T) \quad \text{uniformly in } T \geq 1$$

(where zeroes are counted according to multiplicity).

(iv) If at least one root of $\Lambda(c, \gamma, s)$ lies on the line $\sigma = e_{F}$, then there exist $\mu \in (0, 1), \omega = \omega(c) \geq 0, a = a(c; \gamma) \in \mathbb{R}$, and a curve $G_{e, \gamma} : \tau \to (g_{e, \gamma}(\tau), \tau)$, where $g_{e, \gamma}(\tau) : \mathbb{R} \to (e_{F} - \mu, e_{F})$ is a continuous function, satisfying the following three properties.

(a) In the non-latticelike case, $\omega = 0$. In the latticelike case, we can set $g_{e, \gamma}(\tau) = \sigma_{0}$ for any fixed $\sigma_{0} \in (e_{F} - \mu, e_{F})$.
(b) The zeroes of $\Lambda(c; \gamma; s)$ on the line $\{\sigma = e_{F}\}$ are contained in the set $\{e_{F} + i(a + \ell\omega)\}_{\ell \in \mathbb{Z}}$.
(c) The Dirichlet polynomial $\Lambda(c; \gamma; s)$ satisfies the lower bound

$$\Lambda(c, \gamma, s)|_{G_{e, \gamma}(\tau)} \gg 1 \quad \text{for all } \tau,$$

and all its zeroes to the right of $G_{e, \gamma}$ are simple.

The curve $G_{e, \gamma}$ is called a ‘screen’ for the Dirichlet polynomial $\Lambda(c, \gamma, s)$.

(v) Defining the ‘window’ $W_{e, \gamma} := \{ s = \sigma + i\tau; \; \sigma \geq g_{e, \gamma}(\tau) \}$ for the screen $G_{e, \gamma}$ the following two properties are satisfied:

(a) the derivatives of $\Lambda(c, \gamma, s)$ satisfy the property:

$$\Lambda'(c, \gamma, \rho) \gg 1 \quad \text{uniformly over the zeroes } \rho \text{ of } \Lambda \text{ in } W_{e, \gamma};$$

(b) there exist two monotonic unbounded sequences $\{T_{n}\}_{n \geq 1} \subset (0, \infty)$ and $\{T_{-n}\}_{n \geq 1} \subset (-\infty, 0)$ such that:

$$\Lambda(c, \gamma, \sigma + iT_{\pm n}) \gg 1 \quad \text{uniformly in } n \in \mathbb{N} \text{ and } \sigma \geq g_{e, \gamma}(T_{\pm n}).$$

Remark. In particular, in the non-latticelike case, if there is a pole on the line $\{\sigma = e_{F}\}$, then it is unique. In the latticelike case, the lower bounds in Parts iv(c) and v(b) are satisfied even when there is no zero of $\Lambda(c, \gamma, s)$ on the line $\sigma = e_{F}$. The other parts are all vacuously satisfied in this event.

Proof. The first point concerns the proof of Part (iii). This is an immediate consequence of Theorem 3.6 of [20].

Proof of Part (i). Let $s = \sigma + i\tau \in \mathbb{C}$. If $\sigma > e_{F}$, then $|\sum_{j=1}^{r} \gamma_{j} c_{j}^{-s}| \leq \sum_{j=1}^{r} |\gamma_{j} c_{j}^{-s}| = \sum_{j=1}^{r} e_{j}^{-\sigma} < \sum_{j=1}^{r} e_{j}^{-e_{F}} = 1$. It follows that $\Lambda(c; \gamma; s) \neq 0$.

Proof of Part (ii). Assume that $s = e_{F} + i\tau$ is a zero of $\Lambda(c; \gamma; s)$ on the line $\{\sigma = e_{F}\}$. It follows that $1 = |\sum_{j=1}^{r} \gamma_{j} c_{j}^{-s}| \leq \sum_{j=1}^{r} |\gamma_{j} c_{j}^{-s}| = \sum_{j=1}^{r} e_{j}^{-e_{F}} = 1$. Thus, there exist positive numbers $\mu_{1}, \ldots, \mu_{r} > 0$ and a real number $\theta$ such that $\gamma_{j} c_{j}^{-s} = \mu_{j} e^{i\theta}$ for all $j = 1, \ldots, r$. 

\[\square\]
Since each introduce some needed notations. Thus, \( B = 0 \) to define the twisted fractal zeta functions, we must first fix an orthonormal basis \( \lambda \). The vectors \( \lambda \) diagonalizable. The vectors \( \lambda \)\( \gamma \)\( \text{and} \) to which Theorems 3.25, 3.26 of [20] do apply. This completes the proof of the lemma.

2.2.2. Twisted fractal zeta functions. Before beginning the discussion here, we first introduce some needed notations.

**Notations:** We set \( N_0 = \mathbb{N} \cup \{0\} \), and for any \( \alpha = (\alpha_1, \ldots, \alpha_n) \in N_0^n \), \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n \), we define

\[
|\alpha| = \sum_i \alpha_i \quad \text{and} \quad \lambda^\alpha := \prod_j \lambda_j^{\alpha_j}.
\]

To define the twisted fractal zeta functions, we must first fix an orthonormal basis \( B = \{g_1, \ldots, g_n\} \) of \( E_C \), the complexification of \( E \), with respect to which each \( T_j \) is diagonalizable. The vectors \( \lambda_j \in (S^1)^n \) are then defined as in Definition 1.

Each element \( m \) of \( F \) is written in the basis \( B \) as a linear combination \( m = \sum_j m_j g_j \). In these coordinates, we define the (a priori formal) twisted fractal zeta functions

\[
\zeta_F(\alpha; s) := \sum_{m \in F} \frac{m^\alpha}{|m|^s} \quad (s = \sigma + i\tau).
\]

The proof of the following lemma then becomes a routine application of the idea from [6] and Lemma 1. In addition, one sees quite clearly, how the knowledge about the zeroes of the Dirichlet polynomial \( \Lambda(c, \lambda^\alpha, s) \) translates into information about the poles of \( \zeta_F(\alpha, s) \).
Lemma 2. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$. The following properties are satisfied for any $\zeta_F(\alpha, s)$:

(i) $s \mapsto \zeta_F(\alpha; s)$ converges absolutely in $\{\sigma > e_F + |\alpha|\}$, has a meromorphic continuation with moderate growth (see Remark below) to the complex plane $\mathbb{C}$ and its polar locus is a subset of $\bigcup_{\alpha \in \mathbb{N}_0^n} \bigcup_{k \in \mathbb{N}} \left\{ s + |\alpha| - k : \sum_{j=1}^r \lambda_j^\alpha e_j^{-s} = 1 \right\}$;

(ii) for any $\eta \in (0, 1)$, the function $s \mapsto \tilde{\zeta}_F(\alpha; s) := (\sum_{j=1}^r \lambda_j^\alpha e_j^{-s} - 1)\zeta_F(\alpha; s)$ is holomorphic on the half plane $\{\sigma > e_F + |\alpha| - \eta\}$ where it satisfies the estimate $|\tilde{\zeta}_F(\alpha; \sigma + i\tau)| \ll 1 + |\tau|^\eta$;

(iii) if $e_F + |\alpha|$ is a pole of $\zeta_F(\alpha; s)$, then it is a simple pole and $\lambda_j^\alpha = 1$ for all $j = 1, \ldots, r$;

(iv) if $\zeta_F(\alpha; s)$ has at least one pole in the line $\sigma = e_F + |\alpha|$, then there exist $\mu \in (0, 1)$, $a = a(c, \alpha) \in \mathbb{R}$, $\omega = \omega(c) \geq 0$, $\delta = \delta(c, \alpha, \mu) \in (0, 1)$ and a ‘screen’ $G_{c, \alpha}: \tau \mapsto (g_{c, \alpha}(\tau), \tau) = (\sigma, \tau)$, where $g_{c, \gamma}: \mathbb{R} \to (e_F + |\alpha| - \mu, e_F + |\alpha|)$, such that:

(a) the poles of $\zeta_F(\alpha; s)$ on the line $\sigma = e_F + |\alpha|$ are contained in the set $\{e_F + |\alpha| + i(a + \ell \omega) : \ell \in \mathbb{Z}\}$. Moreover, $\omega = 0$ if $F$ is non-latticelike;

(b) the poles $\rho$ of $\zeta_F(\alpha; s)$ in the ‘window’ $W_{c, \gamma} := \{s = \sigma + i\tau; \sigma \geq g_{c, \alpha}(\tau)\}$ are simple and the following estimate is satisfied:

$$\text{Res}_{s=\rho} \zeta_F(\alpha, s) \ll 1 + |\Im(\rho)|^\delta;$$

(c) $\zeta_F(\alpha, s)$ has no pole on the screen $G_{c, \alpha}$;

(d) $\zeta_F(\alpha, s) \ll 1 + |3s|^\delta$ uniformly in $s \in G_{c, \alpha}$;

(e) there exist two monotonic unbounded sequences of real numbers $\{T_n\}_{n \geq 1} \subset (0, \infty)$ and $\{T_{-n}\}_{n \geq 1} \subset (-\infty, 0)$ such that $\zeta_F(\alpha; \sigma + iT_{\pm n}) \ll 1 + |T_{\pm n}|^\delta$ uniformly in $n \in \mathbb{N}$ and $\sigma \geq g_{c, \alpha}(T_{\pm n})$.

(v) in the latticelike case. Parts iv(a)–(e) remain true even if $\zeta_F(\alpha, s)$ has no pole on the line $\sigma = e_F + |\alpha|$. Moreover, the screen in this event can be chosen to be a vertical line to the left of $\{\sigma = e_F + |\alpha|\}$;

(vi) in the non-latticelike case, set $\{0 < \lambda_1 < \lambda_2 < \cdots \}$ to denote the set of distinct values of the function $m \in F' \to \|m\|$, and define $b_k(\alpha) := \sum_{\{m \in F' : \|m\| = \lambda_k\}} m^\alpha$ for all $k \geq 1$.

Then there exists a sequence $\{a_k(\alpha)\}_{k \geq 1} \subset (0, \infty)$ such that

(a) $|b_k(\alpha)| \leq a_k(\alpha)$ for all $k \geq 1$;

(b) the Dirichlet series $s \mapsto \sum_{k=1}^\infty \frac{a_k(\alpha)}{\lambda_k^s}$ converges absolutely in the half plane $\{\sigma > e_F + |\alpha|\}$ and has a meromorphic continuation to an open domain containing $\{\sigma \geq e_F + |\alpha|\}$. This meromorphic function has exactly one pole on $s = e_F + |\alpha|$ which is simple.

Remark. A meromorphic function $F(s)$ with polar locus $\mathcal{P}$ has moderate growth on a domain $\mathcal{D} \subset \mathbb{C}$ if there exists $a, b > 0$ such that $\forall \delta > 0$ and $\forall s \in \{d(s, \mathcal{P} \cap \mathcal{D}) \geq \delta\}$, $F(\sigma + i\tau) \ll_{\sigma, \delta} 1 + |\tau|^{a_\sigma + b}$ (see [6]).
Proof of Lemma 2. Part (i) follows from Theorem 1 of [6].
Part (iv) follows from Part (ii) and Lemma 1 Parts (iv) and (v).
Part (v) follows from Part (ii), Lemma 1 Part iv(a), and the Remark following the statement of that lemma.

Proof of Part (ii). We first remark that for $\sigma > e_F + |\alpha|$:  
\[
\zeta_F(\alpha; s) = \sum_{m \in F'} \frac{\prod_{k=1}^{n} m_k^\alpha}{\|m\|^s} = \sum_{m \in F'} \frac{\prod_{k=1}^{n} (m, g_k)^\alpha}{\|m\|^s}
\]
\[
\equiv \sum_{j=1}^{r} \sum_{m \in F'} \frac{\prod_{k=1}^{n} (f_j(m), g_k)^\alpha}{\|f_j(m)\|^s}
\]
\[
\equiv \sum_{j=1}^{r} \sum_{m \in F'} \frac{\prod_{k=1}^{n} (c_j(T_j(m + u_j), g_k))^\alpha}{\|c_jT_j(m + u_j)\|^s}
\]
\[
\equiv \sum_{j=1}^{r} \sum_{m \in F'} \frac{\prod_{k=1}^{n} (c_j\lambda_{k,j}T_j(m + u_j))^\alpha}{\|c_jT_j(m + u_j)\|^s}
\]
\[
\equiv \sum_{j=1}^{r} \sum_{m \in F'} \frac{c_j |\alpha| - s \lambda_j^\alpha (m + u_j)^\alpha}{\|m + u_j\|^s}
\]
where, for two functions $F(s), G(s)$, the notation $F \equiv G$ means that the function $F - G$ is bounded and holomorphic in the halfplane $\{Rs > a\}$ for any real $a$.

It follows that for $\sigma > e_F + |\alpha|$:  
\[
\tilde{\zeta}_F(\alpha; s) = \left( \sum_{j=1}^{r} \lambda_j^\alpha \frac{\alpha |\alpha| - 1}{\lambda_j^\alpha} \right) \zeta_F(\alpha; s)
\]
\[
\equiv \sum_{j=1}^{r} \frac{\lambda_j^\alpha |\alpha| - 1}{\lambda_j^\alpha} \sum_{m \in F'} \left[ \frac{m^\alpha}{\|m\|^s} - \frac{(m + u_j)^\alpha}{\|m + u_j\|^s} \right]. \tag{2.22}
\]

Now let $K$ be any compact subset of $\mathbb{R}$. We then observe that the following estimate is uniform over all $m \in F'$ such that $\|m\| > 1$ and all $s = \sigma + i\tau \in \mathbb{C}$ such that $\sigma \in K$:  
\[
\frac{m^\alpha}{\|m\|^s} - \frac{(m + u_j)^\alpha}{\|m + u_j\|^s} \ll_K (1 + |\tau|) \frac{\|u_j\|}{\|m\|^{|\sigma| + 1}}. \tag{2.23}
\]

We conclude that $s \mapsto \tilde{\zeta}_F(\alpha; s)$ has a holomorphic continuation with moderate growth to the set $\{\sigma > e_F + |\alpha| - 1\}$ on which $|\zeta_F(\alpha; \sigma + i\tau)| \ll 1 + |\tau|$. By using in addition the classical Phragmén–Lindelöf principle (see [24, Theorem 16, p. 123]), we get the growth estimate (2.21) for $\tilde{\zeta}_F(\alpha, s)$.

Proof of Part (iii). Assume that $s = e_F + |\alpha|$ is a pole of $\zeta_F(\alpha; s)$. We conclude from Part (ii) that  
\[
\sum_{j=1}^{r} \lambda_j^\alpha c_j^{-e_F} = 1. \tag{2.24}
\]
Applying [6, Theorem 2], we deduce that  
\[
1 = \left| \sum_{j=1}^{r} \lambda_j^\alpha c_j^{-e_F} \right| \ll \sum_{j=1}^{r} |\lambda_j^\alpha c_j^{-e_F}| = \sum_{j=1}^{r} c_j^{-e_F} = 1.
\]
It follows that $\lambda_j^\alpha/\lambda_i^\alpha \in (0, \infty)$ for all $j = 1, \ldots, r$, and hence $\lambda_j^\alpha = \lambda_i^\alpha$. Relation (2.24) then implies that

$$\lambda_i^\alpha = \lambda_1^\alpha \left( \sum_{j=1}^r c_j^{-s} \right) = \sum_{j=1}^r \lambda_j^\alpha c_j^{-s} = 1.$$ 

We deduce that $\lambda_j^\alpha = 1$ for all $j$.

It follows that

$$\frac{d}{ds} \left( \sum_{j=1}^r \lambda_j^\alpha |j|^{-s} - 1 \right) \bigg|_{s=e_\mathcal{F} + |\alpha|} = - \sum_{j=1}^r c_j^{-e_\mathcal{F}} \ln c_j \neq 0. \quad (2.25)$$

Combining this with the fact that $\tilde{\zeta}_{\mathcal{F}}(\alpha; s)$ is analytic at $s = e_\mathcal{F} + |\alpha|$ then implies that $e_\mathcal{F} + |\alpha|$ is a simple pole of $\zeta_{\mathcal{F}}(\alpha; s)$.

**Proof of (6).** Assume we are in the non-latticelike case. Set $a_k(\alpha) := \sum_{\{m \in \mathcal{F}: \|m\| = \lambda_k\}} |m|^{\|\alpha\|}$ for all $k \geq 1$. Then it is easy to see that for $\sigma > e_\mathcal{F} + |\alpha|$, we have

$$G(s) := \sum_{k=1}^{\infty} \frac{a_k(\alpha)}{\lambda_k^s} = \sum_{m \in \mathcal{F}} \frac{|m|^{\|\alpha\|}}{\|m\|^s} = \sum_{m \in \mathcal{F}} \frac{1}{\|m\|^{s-|\alpha|}} = \zeta(\mathcal{F}, s - |\alpha|),$$

where $\zeta(\mathcal{F}, s) = \sum_{m \in \mathcal{F}} 1/\|m\|^s$. We conclude by applying Theorems 1 and 2 of [6]. This completes the proof of Lemma 2.

**Remark.** The difference between the compatible self-similar case and non-compatible case is most clearly seen in the nature of the Dirichlet polynomial $\Lambda(c, \gamma, s)$, appearing in Part (i) of Lemma 2. In the compatible case, each $|\gamma| = 1$. This, however, need not occur in general. As a result, the main difficulty is to extend the properties of the roots of $\Lambda(c, \gamma, s)$ proved in Lemma 1 to any complex vector $\gamma$.

### 2.3. A multivariate Tauberian theorem

The main result of this subsection is a multivariate Tauberian theorem (see Theorem 8) that suffices for our purposes, though it is hardly the most general result that can be proved. We will apply it to prove each of the theorems stated in §2.1. Since we are unable to find anything remotely comparable in the literature, we believe this result is new.

We start with a function $f : \mathbb{N}^k \to \mathbb{R}_+$ and an unbounded increasing sequence of positive numbers $0 < \lambda_1 < \lambda_2 < \cdots$. We then formally define a multivariate Dirichlet series by setting

$$Z(f, s) := \sum_{n_1, \ldots, n_k \geq 1} f(n_1, \ldots, n_k) \lambda_{n_1}^{-s_1} \cdots \lambda_{n_k}^{-s_k} \quad (s = (s_1, \ldots, s_k) \in \mathbb{C}^k), \quad (2.26)$$

and the averages of the coefficients of $Z(f, s)$:

$$A(f, x) := \sum_{\lambda_{n_1} \leq x_1, \ldots, \lambda_{n_k} \leq x_k} f(n_1, \ldots, n_k) \quad (x = (x_1, \ldots, x_k) \in [1, \infty)^k). \quad (2.27)$$

We next impose conditions about $Z(f, s)$ that are satisfied by each of the series in this article. The properties specified below combine all the features proved in [6] for both the lattice and non-latticelike cases.
Hypotheses assumed satisfied by $Z(f, s)$.

(I) There exist Dirichlet series $Z_{b,j}(s)$, each of the form

$$Z_{b,j}(s) = \sum_{n=1}^{\infty} \frac{a_{b,j}(n)}{\lambda_n^s},$$

and $e_{b,j} > 0$ such that $Z_{b,j}$ is absolutely convergent when $\sigma > e_{b,j}$.

(II) There exist complex numbers $d_b \in \mathbb{C}^*$ such that in the domain $\prod_j \{\sigma_j > D\}$, where $D = \max_{b,j} e_{b,j}$,

$$Z(f, s) = \sum_{b=1}^{B} d_b \prod_{j=1}^{k} Z_{b,j}(s_j). \quad (2.28)$$

Moreover, we assume that there exist $\mu > 0$, $\delta \in (0, 1)$ and $\omega \geq 0$ such that for any $b \in \{1, \ldots, B\}$ and for any $j \in \{1, \ldots, k\}$, the Dirichlet series $Z_{b,j}(s)$ verifies one of the two following assumptions (III) or (IV).

(III) There exists a continuous function $g_{b,j} : \mathbb{R} \rightarrow (e_{b,j} - \mu, e_{b,j})$ so that the following properties are satisfied:

1. $Z_{b,j}(s)$ has a meromorphic continuation to an open neighborhood of the set $W_{b,j} := \{s \in \mathbb{C} : \sigma \geq g_{b,j}(\tau)\}$;

2. any pole $\rho$ of $Z_{b,j}(s)$ in $W_{b,j}$ is simple and

$$\text{Res}_{s=\rho} Z_{b,j}(s) \ll 1 + |\Im \rho|^\delta;$$

3. there exists $\phi_{b,j} \in \mathbb{R}$ such that the poles of $Z_{b,j}(s)$ on the line $\sigma = e_{b,j}$ are elements of $\{e_{b,j} + i(\phi_{b,j} + \ell \omega) : \ell \in \mathbb{Z}\}$;

4. there is no pole on the curve $G_{b,j} := \{g_{b,j}(\tau) + i \tau : \tau \in \mathbb{R}\} (= \partial W_{b,j})$; and

$$N_{b,j}(t) := \text{number of poles } \rho \text{ of } Z_{b,j}(s) \text{ in } W_{b,j} \text{ such that } |\Im \rho| \leq t$$

satisfies the bound:

$$N_{b,j}(t) = O(t) \quad \text{as } t \rightarrow \infty;$$

5. $Z_{b,j}(s) \ll 1 + |\tau|^\delta$ uniformly in $s \in G_{b,j}$;

6. there exist two unbounded monotonic sequences $\{T_n\}_{n \geq 1} \subset (0, \infty)$ and $\{T_{-n}\}_{n \geq 1} \subset (-\infty, 0)$ such that

$$Z_{b,j}(\sigma + iT_{\pm n}) \ll 1 + |T_{\pm n}|^\delta \quad \text{uniformly in } n \in \mathbb{N} \text{ and } \sigma \geq g_{b,j}(T_{\pm n}).$$

or

(IV) There exists a sequence of non-negative number $\{a_{b,j}^*(n)\}_{n \geq 1}$ such that:

1. $|a_{b,j}(n)| \leq a_{b,j}^*(n)$ for all $n \geq 1$;

2. the Dirichlet series $s \mapsto Z_{b,j}(s) := \sum_{n=1}^{\infty} a_{b,j}(n)/\lambda_n^s$ converges absolutely in the open half plane $\{\sigma > e_{b,j}\}$ and has a meromorphic continuation to an open domain containing the closed half plane $\{\sigma \geq e_{b,j}\}$ with no pole on the line $\sigma = e_{b,j}$;

3. the Dirichlet series $s \mapsto Z_{b,j}^*(s) := \sum_{n=1}^{\infty} a_{b,j}^*(n)/\lambda_n^s$ converges absolutely in the open half plane $\{\sigma > e_{b,j}\}$ and has a meromorphic continuation to an open domain containing $\{\sigma \geq e_{b,j}\}$ with a single pole that is simple at $s = e_{b,j}$. 
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follows:

With these data, we then define a polyhedron and a particular subset of its vertex set as

\[ |w(\mathbf{e})| := \sum_j e_{b,j}, \text{ and maximum weight} \]

\[ \omega(Z) := \sup\{|e| : e \in S(Z)\}. \]

With these data, we then define a polyhedron and a particular subset of its vertex set as follows:

(1) \( S_0(Z) := \{ \mathbf{e} \in S(Z); |e| = \omega(Z) \}; \)

(2) \( \Gamma(Z) = \text{convex hull}\{ \mathbf{e} \in S_0(Z); |e| = \omega(Z) \}; \)

(3) \[ \mathcal{V}(Z) = \{ \mathbf{v} : \mathbf{v} \in S_0(Z) \text{ and } \mathbf{v} \text{ is a vertex of } \Gamma(Z) \}. \quad (2.29) \]

We use the preceding properties of \( Z(f, s) \) to describe explicitly the following weighted average:

\[ \mathcal{H}(f, \mathbf{x}) := \sum_{\lambda_n \leq 1, \ldots, \lambda_k \leq x_k} f(n_1, \ldots, n_k) \prod_{j=1}^k \left( 1 - \frac{\lambda_{n_j}}{x_j} \right). \quad (2.30) \]

It is clear that \( \mathcal{H}(f, \mathbf{x}) \leq \mathcal{A}(f, \mathbf{x}) \).

An application of Perron’s ‘weighted’ formula (see [15]) tells us that for any \( \xi > \max\{e_{b,j}\} \),

\[ \mathcal{H}(f, \mathbf{x}) = \frac{1}{(2\pi i)^k} \int_{\xi-i\infty}^{\xi+i\infty} \int_{\xi-i\infty}^{\xi+i\infty} Z(f, s) \prod_{j=1}^k \frac{x_{s_j}^{s_j}}{s_j(s_j + 1)} ds_1 \cdots ds_k. \quad (2.31) \]

Applying (2.28), we first express the left side more explicitly as follows:

\[ \mathcal{H}(f, \mathbf{x}) = \sum_{b=1}^B d_b \prod_{j=1}^k \mathcal{H}_{b,j}(x_{s_j}), \text{ where} \]

\[ \mathcal{H}_{b,j}(x_{s_j}) := \frac{1}{2\pi i} \int_{\xi-i\infty}^{\xi+i\infty} Z_{b,j}(s_j) \frac{x_{s_j}^{s_j}}{s_j(s_j + 1)} ds_j \quad \forall j. \quad (2.32) \]

A consequence of this application is the construction of the following quantity, which will play the role of the expected dominant term for \( \mathcal{H}(f, \mathbf{x}) \). For each \( \mathbf{e} \in \mathcal{V}(Z) \), set

\[ \psi_{\mathbf{e}}(\mathbf{y}) = \sum_{b: \mathbf{e}(b) = \mathbf{e}} \sum_{\mathbf{e} \in \mathbb{Z}^k} c_{\mathbf{e}}(\mathbf{e}; b) e^{i(\phi(b) + \omega \mathbf{e} \cdot \mathbf{y})} \quad (\mathbf{y} = (y_1, \ldots, y_k)), \quad (2.33) \]

where, for all \( \phi(b) = (\phi_{b,1}, \ldots, \phi_{b,k}) \) and for \( \mathbf{e} = (e_1, \ldots, e_k) \),

\[ c_{\mathbf{e}}(\mathbf{e}; b) := \frac{d_b \prod_{j=1}^k \text{Res}_{s_j = e_j + i(\phi_{b,j} + \omega e_j)} Z_{b,j}}{\prod_{j=1}^k (e_j + i(\phi_{b,j} + \omega e_j))(e_j + 1 + i(\phi_{b,j} + \omega e_j))}. \]
We note that (III) Part (2) and (IV) Part (2) of the hypotheses imply that each $\psi_e$ is an absolutely convergent series when $F$ is latticelike. In addition, the following property is satisfied:

$$
\sum_{\{k: e(b)=e\}} \sum_{\ell \in \mathbb{Z}^k} |c_\ell(e(b)|^2 < \infty. \tag{2.34}
$$

If, however, $F$ is non-latticelike, then $\psi_e$ is a finite sum (over the $b$ so that $e(b)=e$) of the products of residues at $e_{b,j}$. Our first result is as follows.

**Theorem 7.** The following asymptotic is satisfied when $x \to (+\infty, \ldots, +\infty)$:

$$
\mathcal{H}(f, x) = \sum_{e \in \mathcal{V}(\mathbb{Z})} x^e \psi_e(\ln x_1, \ldots, \ln x_k) + o \left( \sum_{e \in \mathcal{V}(\mathbb{Z})} x^e \right) + O \left( \sum_{e \in S(\mathbb{Z}) \setminus \mathcal{V}(\mathbb{Z})} x^e \right) \tag{2.35}
$$

In order to prove Theorem 7, we need the following lemma.

**Lemma 3.** Let $b \in \{1, \ldots, B\}$, $j \in \{1, \ldots, k\}$ and $\xi > e_{b,j}$. The function $\mathcal{H}_{b,j}(x)$ defined in (2.32) satisfies (as $x \to \infty$):

$$
\mathcal{H}_{b,j}(f, x_j) = \sum_{\ell_j \in \mathbb{Z}} \text{Res}_{s_j=e_{b,j}+i(\phi_{b,j}+\omega\ell_j)}(Z_{b,j})
\times x^{s_j}_{\xi+\ell_j} \left( x^{\rho_j}_{\xi+\ell_j} \right) + o(x^{e_{b,j}}) \tag{2.36}
$$

In case (III) and $\omega = 0$ in the non-latticelike case respectively in case (IV), the above sum over $\ell_j$ is replaced by a single residue at $s_j = e_{b,j} + i\phi_{b,j}$ respectively 0.

**Proof of Lemma 3.** First case: we assume that assumption (III) holds.

For each $n$, we transport the segment $\{\{\xi, \tau_j\}: T_{-n} \leq \tau_j \leq T_n\}$ to the left and stop along the curve $\mathcal{C}(b, j, n) := \left\{ (S_{b,j}(\tau_j), \tau_j) : T_{-n} \leq \tau_j \leq T_n \right\}$. Along the way, we encounter the possible pole of $Z_{b,j}(s_j)$ at $s_j = e_{b,j} + i(\phi_{b,j} + \omega \ell_j)$ as well as $O(T_n + |T_n|)$ other poles $\rho_{b,j}(r) \in W_{b,j} - \{\sigma_j = e_{b,j}\}$. Denoting the upper resp. lower horizontal segments $\tau_j = T_n$ respectively $\tau_j = T_{-n}$ as $W_j(+n)$ respectively $W_j(-n)$ it is clear that if $\omega \neq 0$, then

$$
\frac{1}{2\pi i} \int_{\xi+iT_{-n}}^{\xi+iT_n} Z_{b,j}(s_j) \frac{x^{s_j}_{\xi+\ell_j}}{s_j(s_j+1)} ds_j
$$

$$
= \sum_{T_{-n} \leq \ell_j \leq T_n} \text{Res}_{s_j=e_{b,j}+i(\phi_{b,j}+\omega\ell_j)}(Z_{b,j})
\times x^{s_j}_{\xi+\ell_j} \left( x^{\rho_{b,j}(r)}_{\xi+\ell_j} \right) + \int_{W_j(+n)-W_j(-n)} Z_{b,j}(s_j) \frac{x^{s_j}_{\xi+\ell_j}}{s_j(s_j+1)} ds_j
$$

$$
+ \int_{\mathcal{C}(b,j,n)} Z_{b,j}(s_j) \frac{x^{s_j}_{\xi+\ell_j}}{s_j(s_j+1)} ds_j \tag{2.37}
$$
If $\omega = 0$ (non-latticelike case), then the above sum over $\ell_j$ is replaced by a single residue at $s_j = e_{b,j} + i\phi_{b,j}$. No change is needed in the three other summands.

Applying Parts (5) and (6) of (III), it is clear that for any $x_j$:

$$
\lim_{n \to +\infty} \int_{W_j(+T_n) - W_j(-T_n)} Z_{b,j}(s_j) \frac{x_j^{s_j}}{s_j(s_j + 1)} \, ds_j = 0,
$$

$$
\lim_{n \to +\infty} \int_{C_{\ell_j}(\beta,n)} Z_{b,j}(s_j) \frac{x_j^{s_j}}{s_j(s_j + 1)} \, ds_j = \int_{C_{\ell_j}(\beta)} Z_{b,j}(s_j) \frac{x_j^{s_j}}{s_j(s_j + 1)} \, ds_j \, \text{exists.}
$$

(2.38)

Furthermore, since $\sigma_j |s_j| < e_{b,j}$ for each $s_j \in C_{b,j}$ an easy application of Lebesgue dominated convergence theorem tells us that

$$
\left| \sum_{\ell_j = e_{b,j} + i(\phi_{b,j} + \omega \ell_j)} \left( e_{b,j} + i(\phi_{b,j} + \omega \ell_j) \right) \frac{x_j^{s_j}}{s_j(s_j + 1)} \right| \text{ converges absolutely.}
$$

Moreover, since $\sigma_j |s_j| < e_{b,j}$ for each $s_j \in C_{b,j}$ we conclude that (2.36) holds if the hypotheses in (III) are all satisfied.

Part (2) of (III) implies

$$
\sum_{\ell_j = e_{b,j} + i(\phi_{b,j} + \omega \ell_j)} \left( e_{b,j} + i(\phi_{b,j} + \omega \ell_j) \right) \frac{x_j^{s_j}}{s_j(s_j + 1)} \text{ converges absolutely.}
$$

The last point to address is the behavior in $x_j$ of the limit as $n \to \infty$ of

$$
\sum_{T-n \leq \Re \rho_{b,j}(r) \leq T_n} \Res_{s_j = \rho_{b,j}(r)} \left( Z_{b,j}(s_j) \frac{x_j^{s_j}}{s_j(s_j + 1)} \right).
$$

We first show that the series obtained by replacing the monomial in $x_j$ by the constant 1 converges absolutely.

By choosing $\mu$ small enough, we can assume that $e_{b,j} - \mu > 0$. Set $\gamma = \frac{1}{2}(e_{b,j} - \mu)$. Part (4) of (III) implies that we have uniformly in $T \gg 1$:

$$
\sum_{|\Im \rho_{b,j}(r)| \leq T} \left| \frac{\Res_{s_j = \rho_{b,j}(r)}(Z_{b,j})}{\rho_{b,j}(r)(\rho_{b,j}(r) + 1)} \right| \ll \sum_{|\Im \rho_{b,j}(r)| \leq T} \frac{1}{(\gamma + |\Im \rho_{b,j}(r)|)^{2-\delta}} = \int_0^T (\gamma + t)^{-2+\delta} \, dN_{b,j}(t)
$$

$$
= (\gamma + T)^{-2+\delta} N_{b,j}(T_n) + (2 - \delta) \int_0^T (\gamma + T)^{-2+\delta} N_{b,j}(t) \, dt \ll T^{-1+\delta} + \int_0^{\infty} t^{-2+\delta} \, dt \ll 1.
$$

As a result, by using the fact that $\Re \rho_{b,j}(r) < e_{b,j}$ for each $r$, an application of the Lebesgue’s dominated convergence theorem (using the Stieltjes integral with respect to $dN_{b,j}(t)$) then shows:

$$
\lim_{n \to +\infty} \sum_{|\Im \rho_{b,j}(r)| \leq T_n} \Res_{s_j = \rho_{b,j}(r)} \left( Z_{b,j}(s_j) \frac{x_j^{s_j}}{s_j(s_j + 1)} \right)
$$

$$
= \sum_{\rho_{b,j}(r)} \Res_{s_j = \rho_{b,j}(r)} \left( Z_{b,j}(s_j) \frac{x_j^{s_j}}{s_j(s_j + 1)} \right)
$$

$$
= o(x_j^{\epsilon_{b,j}}) \quad \text{as } x_j \to \infty.
$$

(2.40)

We conclude that (2.36) holds if the hypotheses in (III) are all satisfied.
Second case: we assume that (IV) holds.

We will prove first that
\begin{equation}
A_b(x_j) := \sum_{\lambda_n \leq x_j} a_{b,j}(n) = o(x_j^{e_{b,j}}) \quad \text{as } x_j \to \infty.
\end{equation}

**Proof of (2.41).** We will deduce this estimate from Delange’s Tauberian theorem [4] by using, in addition, an argument in [18] (see Theorem 1, p. 311).

Set \( Z_{b,j,\text{conj}}(s) := \sum_{n=1}^{\infty} a_{b,j}(n)/\lambda_n^s \). It is easy to see that \( Z_{b,j,\text{conj}}(s) = Z_{b,j}(s)^{x_{b,j}} \). We deduce that the series verifies the same assumptions as \( Z_{b,j}(s) \). It follows that the following two Dirichlet series:
\begin{align*}
Z_{b,j,\Re}(s) &:= \sum_{n=1}^{\infty} \frac{a_{b,j}^*(n) - \Re(a_{b,j}(n))}{\lambda_n^s} = Z_{b,j}^*(s) - \frac{1}{2}(Z_{b,j}(s) + Z_{b,j,\text{conj}}(s)) \\
\text{and} \\
Z_{b,j,\Im}(s) &:= \sum_{n=1}^{\infty} \frac{a_{b,j}^*(n) - \Im(a_{b,j}(n))}{\lambda_n^s} = Z_{b,j}^*(s) - \frac{1}{2}(Z_{b,j}(s) - Z_{b,j,\text{conj}}(s))
\end{align*}

have non-negative coefficients, and also satisfy the same assumptions as \( Z_{b,j}(s) \). Moreover, each series has the same residue at the pole \( s = e_{b,j} \).

We next define \( R := \text{Res}_{s=e_{b,j}} Z_{b,j}(s)/e_{b,j} \). Delange’s Tauberian theorem implies then that
\begin{align*}
\sum_{\lambda_n \leq x_j} a_{b,j}(n) - \Re(a_{b,j}(n)) &= R x_j^{e_{b,j}} + o(x_j^{e_{b,j}}), \\
\sum_{\lambda_n \leq x_j} a_{b,j}(n) - \Im(a_{b,j}(n)) &= R x_j^{e_{b,j}} + o(x_j^{e_{b,j}}), \\
\sum_{\lambda_n \leq x_j} a_{b,j}^*(n) &= R x_j^{e_{b,j}} + o(x_j^{e_{b,j}}).
\end{align*}

It follows that
\begin{align*}
\sum_{\lambda_n \leq x_j} \Re(a_{b,j}(n)) = o(x_j^{e_{b,j}}) \quad \text{and} \quad \sum_{\lambda_n \leq x_j} \Im(a_{b,j}(n)) = o(x_j^{e_{b,j}}).
\end{align*}

This completes the proof of (2.41).

We now conclude from (2.41) that
\begin{align*}
\mathcal{H}_{b,j}(f, x_j) &= \sum_{\lambda_n \leq x_j} a_{b,j}(n) \left(1 - \frac{\lambda_n}{x_j}\right) \\
&= x_j^{-1} \int_0^{x_j} A_b(t) \, dt = o(x_j^{e_{b,j}}) \quad \text{as } x_j \to \infty.
\end{align*}

Thus, (2.36) is also satisfied in case (IV). This finishes the proof of Lemma 3. \( \square \)
Proof of Theorem 7. It follows from (2.31) and (2.36) that when \( x \to (+\infty, \ldots, +\infty) \), we have

\[
\mathcal{H}(f, x) = \sum_{b=1}^{B} d_b \prod_{j=1}^{k} \left( \sum_{\ell_j \in \mathbb{Z}} \text{Res}_{s_j = e_{b,j} + i(\phi_{b,j} + \omega \ell_j)} (Z_{b,j}) \right)
\times \frac{x^{e_{b,j} + i(\phi_{b,j} + \omega \ell_j)}}{(e_{b,j} + i(\phi_{b,j} + \omega \ell_j))(1 + e_{b,j} + i(\phi_{b,j} + \omega \ell_j)) + o(x^{e_{b,j}})}
\]

\[
= \sum_{b=1}^{B} d_b \left\{ \sum_{(t_1, \ldots, t_k) \in \mathbb{Z}^k} x^{e(b) + i \phi(b)} \times \prod_{j=1}^{k} \left( \text{Res}_{s_j = e_{b,j} + i(\phi_{b,j} + \omega \ell_j)} (Z_{b,j}) \right) \frac{x^{i \omega \ell_j}}{(e_{b,j} + i(\phi_{b,j} + \omega \ell_j))(1 + e_{b,j} + i(\phi_{b,j} + \omega \ell_j)) + o(x^{e(b)})} \right\}.
\]

Rearranging the terms and making evident simplifications shows that \( \mathcal{H}(f, x) \) satisfies the asserted asymptotic when \( x \to (+\infty, \ldots, +\infty) \). In particular, the factor of \( x^{e(b) + i \phi(b)} \) is an absolutely convergent Fourier series in the latticelike case. In the non-latticelike case, the factor of \( x^{e(b) + i \phi(b)} \) is a linear combination (over \( b \)) such that \( e(b) = e \) of \( d_b \) multiplied by an iterated residue at \( s = e(b) \). Each such factor is denoted by \( \psi_e \).

The next point applies this to derive a non-trivial lower bound for

\[
\mathcal{A}(f, x) := \mathcal{A}(f, (x_1, \ldots, x_k)) = \sum_{\lambda_{n_1} \leq x, \ldots, \lambda_{n_k} \leq x} f(n_1, \ldots, n_k).
\]

This is possible when, for any vertex \( e \in \mathcal{V}(Z) \), the iterated residue of \( Z(= Z(f, s)) \) at the point \( e \) is non-vanishing. We denote/define this quantity by

\[
\text{Reg } Z(f, e) := \sum_{b=1}^{B} d_b \prod_{j=1}^{k} \text{Res}_{s_j = e_{b,j}} Z_{b,j}(s_j).
\]

More precisely, we will derive from Theorem 7 the following Tauberian theorem.

**Theorem 8.** Assume that for any vertex \( e \in \mathcal{V}(Z) \) \((Z = Z(f, s))\)

\[
\text{Reg } Z(f, e) \neq 0.
\]

Then, for any \( \varepsilon > 0 \), there exists \( C(\varepsilon) > 0 \) such that

\[
\mathcal{A}(f, x) \geq C(\varepsilon) x^{\omega(Z) - \varepsilon} \text{ uniformly in } x \geq 1.
\]

**Remark.** In the following, we abbreviate (2.45) by writing \( \mathcal{A}(f, x) \gg_{\varepsilon} x^{\omega(Z) - \varepsilon} \) when \( x \to \infty \).

**Proof.** First, we remark that we can assume without loss of generality that for all \( b = 1, \ldots, B \) and all \( j = 1, \ldots, k \), we have

\[
\phi_{b,j} = 0 \text{ or } \phi_{b,j} \not\in \mathbb{Q} \omega.
\]

Given \( e \in \mathcal{V}(Z) \) such that (2.44) holds, it follows that the series \( \psi_e(y) \) defined in (2.33) is a non-zero absolutely convergent Fourier series. Using \( \psi_e \), we now construct a non-zero almost periodic function in a single variable with positive mean for any sufficiently small \( \varepsilon > 0 \).
We first choose any \( \varepsilon \). Next, we remark that there exists \( \theta = (\theta_1, \ldots, \theta_k) \in [1, 1 + \varepsilon)^k \) such that

\[
\langle e, \theta \rangle \neq \langle e', \theta \rangle \quad \forall e \neq e' \in S(Z) \quad \text{and} \quad \\
\langle \phi(b) + \omega \ell, \theta \rangle \neq 0 \quad i f \ell \in \mathbb{Z}^k \setminus \{(0, \ldots, 0)\} \text{ or } \phi(b) \neq (0, \ldots, 0).
\]

(2.47)

It suffices to choose a vector \( \theta \) each of whose coordinates is transcendental over \( \mathbb{Q}(\omega, \{\phi(b)\}_{b \in B}) \) and belongs to \( [1, 1 + \varepsilon) \), such that the vector lies in the open subset defined by the following conditions:

\[
\langle e - e', \theta \rangle \neq 0 \quad \text{for all } e \neq e' \in S(Z).
\]

Clearly, such \( \theta \) exist. It then follows that there exists a unique \( e \in \mathcal{V}(Z) \) such that

\[
\langle e, \theta \rangle > \langle e', \theta \rangle \quad \forall e' \in S(Z) \setminus \{e\}.
\]

(2.48)

Setting \( x_j = x^{\theta_j} \), \( \forall j = 1, \ldots, k \) in Theorem 7, we conclude

\[
\mathcal{A}(f, x^{1+\varepsilon}) \geq \mathcal{A}(f, x^{\theta_1}, \ldots, x^{\theta_k}) \geq \mathcal{H}(f, x^{\theta_1}, \ldots, x^{\theta_k}) = x^{\langle e, \theta \rangle} [\psi_e(\theta_1 \ln x, \ldots, \theta_k \ln x) + o(1)].
\]

(2.49)

We now define the function of a single real variable \( y \):

\[
\psi_{e, \theta}(y) := \psi_e(\theta_1 y, \ldots, \theta_k y) = \sum_{\{b: e(b) = e\}} \sum_{\ell \in \mathbb{Z}^k} c_{e}(\ell; b) e^{i(\phi(b) + \omega \ell, \theta)y}.
\]

Denoting the distinct values \( (\phi(b) + \omega \ell, \theta) \) as \( (\eta_n)_{n \in \mathbb{Z}} \), it follows that

\[
\eta_n \neq \eta_m \quad \text{if } n \neq m.
\]

As a result, when we rearrange the terms and write \( \psi_{e, \theta} \) as a series summed over \( n \)

\[
\psi_{e, \theta}(y) = \sum_{n \in \mathbb{Z}} C_n e^{i\eta_n y}
\]

the algebraic independence property in (2.47) insures three essential properties.

(1) There exists a unique integer \( n_0 \) such that

\[
C_{n_0} = \sum_{\{b: e(b) = e\}} \frac{d_{b} \prod_{j=1}^{k} \text{Res}_{s=\varepsilon} \varepsilon Z_{b,j}}{\prod_{j=1}^{k} \varepsilon_j (e_j + 1)} = \frac{\text{Reg} Z(f, e)}{\prod_{j=1}^{k} e_j (e_j + 1)} \neq 0.
\]

(2) The series \( \psi_{e, \theta}(y) \) converges absolutely and is non-zero.

(3) The function \( |\psi_{e, \theta}|^2 \) has a non-zero mean value that does not depend upon \( \theta \).

\( \square \)

Remark. Part (1) is clear from (2.47). Indeed, the independence property implies that each \( C_0 \neq 0 \) whenever \( C_n = c_{e}(\ell, b) \neq 0 \). Part (2) follows from the proof of Theorem 7. As a result, we know that \( \psi_{e, \theta} \) is an almost periodic function (see [2]) whose Fourier series coefficients are precisely the \( C_n \). This allows us to apply Parseval’s theorem for almost periodic functions, since (2.34) tells us that the sum of the \( |C_n|^2 \) converges (and is necessarily positive and independent of \( \theta \)).

This allows us to apply Claim 4 of [6] to \( \psi_{e, \theta}(y) \). We conclude that there exists an unbounded increasing sequence \( \{x_\varepsilon(m)\}_m \) of positive numbers satisfying these two properties:

\[
x_\varepsilon(m) \leq x_\varepsilon(m + 1) \leq x_\varepsilon(m)^{1+\varepsilon}
\]
and
\[
\psi_{\varepsilon, \theta}(\ln x_{\varepsilon}(m)) \gg 1 \quad \text{it uniformly in } m \geq 1 \text{ and } \theta.
\]

This and (2.49) imply
\[
A(f, x_{\varepsilon}(m)^{1+\varepsilon}) \gg x_{\varepsilon}(m)^{|\varepsilon|} = x_{\varepsilon}(m)^{\omega(Z)} \quad \text{uniformly in } m.
\] (2.50)

We now let \( x \in [x_{\varepsilon}(1), \infty) \). There exists a unique \( m \in \mathbb{N} \) such that
\[
x_{\varepsilon}(m)^{1+\varepsilon} \leq x < x_{\varepsilon}(m+1)^{1+\varepsilon} \leq x_{\varepsilon}(m)^{(1+\varepsilon)^2} \leq x_{\varepsilon}(m)^{1+3\varepsilon}.
\]

We deduce then from (2.50) that
\[
A(f, x) \gg A(f, x_{\varepsilon}(m)^{1+\varepsilon}) \gg x_{\varepsilon}(m)^{\omega(Z)} \gg x^{\omega(Z)/(1+3\varepsilon)} \gg x^{\omega(Z)-\varepsilon}.
\]

As a result, we conclude that \( A(f, x) \) satisfies the asserted lower bound when \( x \to +\infty \).

3. Distinct distances, angles, and rooted configurations

3.1. Distinct distances

Proof of Theorem 1. Recall that the Dirichlet series for this problem is defined as follows (setting \( s = (s_1, s_2) \)):
\[
\zeta_2(s) := \sum_{(m_1, m_2) \in (\mathcal{F} - \{0\})^2} \frac{\|m_1 - m_2\|^2}{\|m_1\|^{s_1} \|m_2\|^{s_2}}.
\] (3.1)

It is not difficult to see that the finiteness of the upper Minkowski dimension implies that \( \zeta_2(s) \) converges absolutely in the domain \( \sigma_i := \Re s_i > e_F + 2 \quad \forall i \) (see [6, § 2]). It can therefore be represented in a form more clearly connected to distinct distances. Denoting

- Distinct values of the function \( m \in \mathcal{F} \longrightarrow \|m\| := \{0 < \lambda_1 < \lambda_2 < \cdots\} \);
- Distance set := \( \{\|m_1 - m_2\| : m_1, m_2 \in \mathcal{F}\} \);
- the Distinct distance set := \( D_i = \{\rho_j\} \);

and setting, for any pair \( k = (k_1, k_2) \) of subscripts of the \( \lambda \) and \( \rho \in D_i \)
\[
\begin{align*}
\rho_k &= \sum_{(m_1, m_2) \in (\mathcal{F} - \{0\})^2 : \|m_i\| = \lambda_{k_i} \quad \forall i} \|m_1 - m_2\|^2, \\
N_{\rho}(k) &= \#\{(m_1, m_2) \in (\mathcal{F})^2 : \|m_i\| = \lambda_{k_i} \quad \forall i \text{ and } \|m_1 - m_2\| = \rho\}.
\end{align*}
\]

it is clear that in \( \{s : \sigma_i > e_F + 2 \quad \forall i\} \) we have the identity
\[
\zeta_2(\mathcal{F}, s) = \sum_{k} \frac{b_k}{\lambda_{k_1}^{s_1} \lambda_{k_2}^{s_2}} = \sum_{k} \frac{\sum_{\rho \in D_i} N_{\rho}(k) \rho^2}{\lambda_{k_1}^{s_1} \lambda_{k_2}^{s_2}}.
\]

Remark. We subsequently adopt the notations of § 2.3 by setting
\[
f_1(k) = b_k; \quad \zeta_2(\mathcal{F}, s) = Z(f_1, s).
\]

Given any vector \( x = (x_1, x_2) \in (0, \infty)^2 \), the averages, analogs to those defined in (1.2), that interest us are as follows:
\[
A(f_1, x) = \sum_{\{k : \lambda_{k_i} < x_i \quad \forall i\}} b_k.
\] (3.2)


Setting (see Definition 2)
\[D_i(x) := \{ \rho \in D_i : \exists (m_1, m_2) \in F(x_1) \times F(x_2) \text{ such that } \rho = \|m_1 - m_2\| \},\]
\[N_\rho(x) = \#\{(m_1, m_2) \in F(x_1) \times F(x_2) : \|m_1 - m_2\| = \rho \},\]
we know
\[A(f_1, x) = \sum_{\rho \in D_i(x)} N_\rho(x) \cdot \rho^2. \tag{3.3}\]

Defining
\[M(x) := \#D_i(x), \quad \text{and setting } D_i(x) = \{ \rho_1 < \rho_2 < \cdots < \rho_{M(x)} \},\]
the Cauchy–Schwartz bound implies
\[A(f_1, x) = \sum_{\rho \in D_i(x)} N_\rho(x) \cdot \rho^2 \leq \|(\rho_1^2, \ldots, \rho_{M(x)}^2)\| \cdot \|(N_{\rho_1}(x), \ldots, N_{\rho_{M(x)}}(x))\|. \tag{3.4}\]

Since \(\|m_1 - m_2\|^2 = \|m_1\|^2 + \|m_2\|^2 - 2(m_1, m_2)\), it follows that
\[Z(f_1, s) = \zeta(F, s_1 - 1)\zeta(F, s_2) + \zeta(F, s_1)\zeta(F, s_2 - 1) + \zeta^*(s),\]
where
\[\zeta^*(s) = -2\sum_{\|m_1, m_2\|} \frac{\langle m_1, m_2 \rangle}{\|m_1\|^s \|m_2\|^s}.\]

Setting \(Z = Z(f_1, s)\) for simplicity, the polygon \(\Gamma(Z)\), defined in § 2.3, has two vertices
\[v_1 = (D_F, e_F), \quad v_2 = (e_F, D_F)\]
which implies \(\omega(Z) = 2(e_F + 1)\).

In addition, any pole of \(\zeta^*\) lies either in the interior of the segment \([v_1, v_2]\) or strictly below this segment (see [6, §2]). It is also clear from the above expression for \(Z(f_1, s)\) that
\[\text{Reg } Z(f_1, v_1) \neq 0 \quad \text{and } \text{Reg } Z(f_1, v_2) \neq 0. \tag{3.5}\]

We are now justified in applying Theorem 8. This gives a lower bound for \(A(f_1, x)\) (see (2.42)) as follows:
\[\forall \eta > 0 \quad x^{\omega(Z) - \eta} = x^2(e_F + 1)^{-\eta} \ll_{\eta} A(f_1, x) \quad \text{when } x \to \infty. \tag{3.6}\]

Note: Restricting from now on the vector \(x\) to the point \((x, x)\), we also write \(M(x), N_\rho(x)\) in place of \(M(x, x), N_\rho(x, x)\). It is then clear that \(\text{Dis}_F(x) = M(x)\).

There is an evident upper bound for the length of the vector in (3.4) whose components are \(\rho^2\):
\[\|(\rho_1^2, \ldots, \rho_{M(x)}^2)\| \ll x^2 \cdot \sqrt{M(x)}. \tag{3.7}\]

We now use the hypothesis that \(F \subset \mathbb{Z}^n\). This allows us to bound the other vector’s length in (3.4) by applying classical work that described the number of representations of \(\rho^2\) as a sum of \(n\) squares of integers [12]. Although it is well known that precise asymptotics exist for this representation number when \(n \geq 5\), we only need upper bounds valid for each \(n \geq 2\). If \(n = 2\), then the upper bound is elementary. If \(n = 3, 4\), then the bound is due to Bateman [1].

We first use the estimate that follows from the definition of \(e_F\) in Definition 2:
\[\#\{m_2 \in F(x)\} = O_{\theta_1}(x^{e_F + \theta_1}). \tag{3.8}\]

For any \(\xi \in F(x)\), the estimate we need for any \(n \geq 2\) is as follows:
\[\#\{m_1 \in F(x) : \|m_1 - \xi\|^2 = \rho^2\} \ll_{\theta_2} \rho^{n-2+\theta_2} \quad \text{uniformly in } \xi. \tag{3.9}\]
Multiplying together (3.8) and (3.9) implies
\[ N_\rho(x) = O_{\theta_1, \theta_2}(x|e_\rho + n - 2 + \theta_1 + \theta_2|). \]

Combining this with (3.6) and (3.7), we obtain
\[ x^{2(e_\rho + 1) - \eta} \ll_{\eta} A(f_1, x) \ll_{\theta_1, \theta_2} x^{|e_\rho + n + \theta_1 + \theta_2|} . M(x) \quad \text{when } x \to +\infty. \]  

(3.10)

We conclude that if \( e_\rho > n - 2 \), then for any \( \varepsilon_1 > 0 \) so that \( e_\rho > n - 2 + \varepsilon_1 \), we can find \( \eta, \theta_1, \theta_2 > 0 \) (depending only upon \( \varepsilon_1 \)) so that
\[ [2(e_\rho + 1)] - \eta - [e_\rho + n + \theta_1 + \theta_2] > e_\rho - (n - 2) - \varepsilon_1 > 0. \]  

(3.11)

This now gives the non-trivial lower bound
\[ M(x) \gg_{\varepsilon_1} x^{e_\rho - n + 2 - \varepsilon_1}. \]  

(3.12)

To find a lower bound in terms of \( \#\mathcal{F}(x) \), given that (3.12) holds, it suffices to use the upper bound
\[ \#\mathcal{F}(x) \ll_{\varepsilon_2} x^{e_\rho + \varepsilon_2} \quad \text{as } x \to \infty, \]  

(3.13)

which then gives us the lower bound
\[ M(x) \gg_{\varepsilon_1, \varepsilon_2} [\#\mathcal{F}]^{(e_\rho - n + 2 - \varepsilon_1)/(e_\rho + \varepsilon_2)}. \]  

(3.14)

As a result, given any \( \varepsilon > 0 \), we now choose \( \varepsilon_1, \varepsilon_2 > 0 \) so that \( e_\rho - n + 2 > \varepsilon_1 \) and
\[ \frac{e_\rho - n + 2 - \varepsilon_1}{e_\rho + \varepsilon_2} > 1 - \frac{n - 2}{e_\rho} - \varepsilon. \]

This tells us that
\[ \text{Dis}_\mathcal{F}(x) = M(x) \gg_{\varepsilon} [\#\mathcal{F}]^{1 - (n - 2)/e_\rho - \varepsilon} \quad \text{as } x \to \infty, \]

which completes the proof of Theorem 1.

\[ \square \]

Remarks. (1) If \( n = 2 \), our lower bound of \( \gg_{\varepsilon} [\#\mathcal{F}(t)]^{1 - \varepsilon} \) for all \( t \) sufficiently large, which holds for any compatible self-similar subset of \( \mathbb{Z}^2 \) (recall that \( e_\rho > 0 \) follows from Definition 3), is only a bit worse than what follows in general from the celebrated work of Guth–Katz [13]. Moreover, we note that when \( n > 2 \) and \( e_\rho \) satisfies
\[ n \cdot \frac{n^2 - 4}{n^2 - 1} < e_\rho \leq n, \]

the exponent of \( \#\mathcal{F}(t) \) is larger than that obtained in [22], which was proved in much greater generality. The point is that if \( e_\rho \) is in the above interval, then the asymptotic result we can prove gives a larger exponent than that which follows from [22].

(2) The reader will have already observed that there would appear to be room for improving this result provided one could be more precise about the location of points in \( \mathcal{F}(x) \) that actually belong to the sphere centered at a point in \( \mathcal{F}(x) \) and of radius \( \rho \). In the above estimation, via the circle method, there is nothing that helps tell us which of the points on any such sphere actually belong to \( \mathcal{F}(x) \). A better upper bound for (3.9) would surely help improve the lower bounds for \( e_\rho \) and \( \text{Dis}_\mathcal{F}(x) \).

3.2. Angles

Here we prove Theorem 2. To do this efficiently, we first introduce some needed notations and then prove two preliminary lemmas.

We use the standard basis \( e_1, \ldots, e_n \) of the Euclidean space \( E \) to determine the coordinates of any point \( m \in \mathcal{F} \), and any monomial \( m^\alpha \). In contrast to the discussion in §2.2.2, the twisted
zeta functions we use in this subsection are all understood to use the coordinates of any \( m \in \mathcal{F}' \) with respect to \( e_1, \ldots, e_n \).

Using this basis and the coordinates with respect to the basis, we define (formally) the ‘angle zeta function’ \( \zeta_{\text{ang}}(\mathcal{F}; s) \) by

\[
\zeta_{\text{ang}}(\mathcal{F}, s) := \sum_{m_1, m_2 \in \mathcal{F}'} \frac{(m_1/\|m_1\|, m_2/\|m_2\|)^2}{\|m_1\|^s_1 \|m_2\|^s_2} \quad (s = (s_1, s_2)).
\]

Thus, the coefficients are squares of the cosines of the angles \( \theta(m_1, m_2) \) (see (2.7)).

The first of the two lemmas is analytic in nature.

**Lemma 4.** \( \zeta_{\text{ang}}(\mathcal{F}, s) \) satisfies the following four properties:

(i) the formal series defining \( \zeta_{\text{ang}}(\mathcal{F}, s) \) converges absolutely in the domain \( \{\sigma_1 > e_\mathcal{F}\} \cap \{\sigma_2 > e_\mathcal{F}\} \);

(ii) the zeta function \( \zeta_{\text{ang}}(s) \), defined by this series, satisfies the Hypotheses (I), (II), and either (III) or (IV), which are needed to apply Theorem 7;

(iii) the vertex set (see (2.29)) of \( \zeta_{\text{ang}}(\mathcal{F}, s) \) is a singleton set \( \mathcal{V}(\zeta_{\text{ang}}) = \{(e_\mathcal{F}, e_\mathcal{F})\} \), and

\[
\omega(Z) = 2e_\mathcal{F} \quad (Z = \zeta_{\text{ang}}(\mathcal{F}, s));
\]

(iv) \( \text{Reg} \zeta_{\text{ang}}(\mathcal{F}, (e_\mathcal{F}, e_\mathcal{F})) = \lim_{s \to (e_\mathcal{F}, e_\mathcal{F})} (s_1 - e_\mathcal{F})(s_2 - e_\mathcal{F}) \zeta_{\text{ang}}(\mathcal{F}, s) \neq 0. \)

**Proof of Lemma 4.** Part (i) follows from the inequality \(|\langle m_1, m_2 \rangle| \leq \|m_1\| \|m_2\| \).

Moreover, it is easy to see that for any \( s \in \{\sigma_1 > e_\mathcal{F}\} \cap \{\sigma_2 > e_\mathcal{F}\} \), we have

\[
\zeta_{\text{ang}}(\mathcal{F}; s) = \sum_{i=1}^{n} \zeta_{\mathcal{F},0}(2e_i; s_1 + 2) \zeta_{\mathcal{F},0}(2e_i; s_2 + 2) + 2 \sum_{1 \leq i < j \leq n} \zeta_{\mathcal{F},0}(e_i + e_j; s_1 + 2) \zeta_{\mathcal{F},0}(e_i + e_j; s_2 + 2),
\]

where \( \zeta_{\mathcal{F},0}(\alpha, s) \) denotes the twisted zeta function when the monomial \( m^\alpha \) is written in the coordinates determined by the standard unit basis \( \{e_1, \ldots, e_n\} \) of \( \mathbb{R}^n \).

Since coordinate changes are defined by linear functions, whenever \( \alpha \in \mathbb{N}_0^n \) satisfies \( |\alpha| = 2 \), \( \zeta_{\mathcal{F},0}(\alpha, s) \) equals a linear combination of the twisted zeta functions \( \zeta_{\mathcal{F}}(\beta, s) \) with \( |\beta| = 2 \) that were defined in § 2.2.2. As a result, Parts (ii) and (iii) follow from Lemma 2.

Moreover, Lemma 2 implies that the limit defining \( \text{Reg} \zeta_{\text{ang}}(\mathcal{F}, (e_\mathcal{F}, e_\mathcal{F})) \) exists and can be computed as follows:

\[
\text{Reg} \zeta_{\text{ang}}(\mathcal{F}, (e_\mathcal{F}, e_\mathcal{F})) = \lim_{\sigma \to e_\mathcal{F}} (\sigma - e_\mathcal{F})^2 \zeta_{\text{ang}}((\mathcal{F}, (\sigma, \sigma))).
\]

We can now restrict \( s \) to the real vector \( (\sigma, \sigma) \). It then follows from (3.16) that for any \( \sigma > e_\mathcal{F} \), we have

\[
\zeta_{\text{ang}}((\mathcal{F}, (\sigma, \sigma))) = \sum_{i=1}^{n} \zeta_{\mathcal{F},0}^2(2e_i, \sigma + 2) + 2 \sum_{1 \leq i < j \leq n} \zeta_{\mathcal{F},0}^2(e_i + e_j, \sigma + 2)
\]

\[
\geq \sum_{i=1}^{n} \zeta_{\mathcal{F},0}^2(2e_i, \sigma + 2) \geq \frac{1}{n} \left( \sum_{i=1}^{n} \zeta_{\mathcal{F},0}(2e_i, \sigma + 2) \right)^2
\]

(by an application of Cauchy–Schwarz)

\[
\geq \frac{1}{n} \left( \sum_{m \in \mathcal{F}'} \frac{1}{\|m\|^\sigma} \right)^2 = \frac{1}{n} \zeta_{\mathcal{F}}^2(\sigma).
\]
We can then apply Theorem 2 of [6], which tells us that \( \zeta_F(s) \) has a simple pole at \( e_F \). Thus, by letting \( \sigma \rightarrow e_F \), we conclude that
\[
\text{Reg} \zeta_{\text{ang}}(F; (e_F, e_F)) \geq \frac{1}{n} (\text{Res}_{s=e_F} \zeta_F)^2 > 0.
\]
This finishes the proof of Lemma 4.

To state the next lemma, we first define, for any \( \theta \in [0, 2\pi) \) and \( t \in F' \):
\[
N_\theta(t, x) = \# \{ m_1 \in F'(x) : \cos^2(\theta(m_1, t)) = \cos^2(\theta) \},
\]
\[
N_\theta(x) = \sum_{t \in F'(x)} N_\theta(t, x).
\]

**Lemma 5.** Assume that \( n \geq 4 \). Then for any \( \varepsilon > 0 \), we have
\[
N_\theta(x) \ll_{\varepsilon} x^{e_F + n - 2 + \varepsilon} \quad \text{uniformly in } \theta \in [0, 2\pi) \setminus \{ \pi/2, 3\pi/2 \} \text{ and } x \geq 1. \quad (3.18)
\]

**Proof of Lemma 5.** Let \( \theta \neq \pi/2, 3\pi/2 \in [0, 2\pi) \). Since \( m_1, m_2 \in F \) implies \( (m_1, m_2)^2, ||m_1||^2, ||m_2||^2 \in \mathbb{Z} \), any \( \theta \) so that \( \theta = \theta(m_1, m_2) \) must satisfy the property that \( \cos^2(\theta) \in \mathbb{Q}^\ast \).

So there exists \( a, b \in \mathbb{Z} \), such that \( (a, b) = 1 \) and \( \cos^2(\theta) = a/b \).

We now choose any \( t \in F'(x) \) and fix it throughout the following discussion. We next define the quadratic form with integer coefficients \( Q_t \) by
\[
Q_t(y) := b(y, t)^2 - a\|t\| \|y\|^2.
\]
It is then clear that
\[
\{ m_1 \in F'(x) : \cos^2(\theta(m_1, t)) = \cos^2(\theta) \} \subset \{ y \in \mathbb{Z}^n : \|y\| \leq x \text{ and } Q_t(y) = 0 \}.
\]

As a result, it suffices to bound the number of points in the subset on the right uniformly in \( t \) and all \( x \geq 1 \). A standard property about integer points on quadric hypersurfaces is that the rank of \( Q_t \) helps to do this even when the form is singular.

A standard calculation shows that the singular locus \( \text{Sing}(Q_t) \) of the quadratic \( \{Q_t(y) = 0\} \) is given by
\[
\text{Sing}(Q_t) = \{ y : (y, t) \parallel t \parallel ^2 y_i \parallel t \parallel ^2 y_i \forall i = 1, \ldots, n \}.
\]
We distinguish two cases.

(i) \( \cos(\theta) \neq \pm 1 \):
In this event,
\[
y \in \text{Sing}(Q_t) \quad \text{implies} \quad \sum_{i=1}^n (y, t) t_i^2 = \cos^2(\theta) \|t\| ^2 t_i y_i.
\]
As a result, summing over \( i \) we get \( (y, t) \|t\|^2 = \cos^2(\theta) \|t\|^2 \langle y, t \rangle \). Since \( 0 < \cos^2(\theta) < 1 \), we conclude that \( (y, t) = 0 \), which implies \( y = 0 \) since \( t \neq 0 \). Thus, \( \text{dim} \text{Sing}(Q_t) = 0 \).

(ii) \( \cos(\theta) = \pm 1 \):
In this event, \( y \in \text{Sing}(Q_t) \) if and only if \( y_i = (t_i/\|t\|) \langle y, t \rangle \forall i = 1, \ldots, n \).

After permutation, we can assume, without loss of generality, that
\[
t_1 = \cdots = t_u = 0 \text{ and } t_i \neq 0 \text{ for all } i = u + 1, \ldots, n.
\]
A point in \( \text{Sing}(Q_t) \) also must lie in \( \{Q_t = 0\} \). This implies that each \( i \in [u + 1, n) \),
\[
y \in \text{Sing}(Q_t) \text{ if and only if } y_1 = \cdots = y_u = 0 \text{ and } y_i = \frac{t_i}{t_n} y_n \forall i < n,
\]
from which it is clear that \( \text{dim} \text{Sing}(Q_t) = 1 \). Thus, \( \text{dim} \text{Sing}(Q_t) \leq 1 \) for any \( t \).
Since \( \text{Rank} \, Q_t \geq \text{codimSing}(Q_t) \), we conclude that
\[
\text{Rank} \, Q_t \geq n - 1 \geq 3 \quad \text{uniformly in } t \in \mathcal{F}' \text{ and } \theta \in [0, 2\pi) \setminus \{\pi/2, 3\pi/2\}. \quad (3.19)
\]

It follows in particular that \( Q_t \) is irreducible over \( \mathbb{Q} \). Indeed, if \( Q_t \) were reducible over \( \mathbb{Q} \), then there would exist two (not necessarily distinct) linear forms \( g_1(x) \) and \( g_2(x) \) over \( \mathbb{Q} \) such that \( Q_t(x) = g_1(x)g_2(x) \). If \( g_1 \) and \( g_2 \) were dependent, then \( \dim \text{Sing}(Q_t) = n - 1 \), which is not possible when \( n \geq 4 \). If \( g_1, g_2 \) were independent, then the intersection \( \{g_1 = 0\} \cap \{g_2 = 0\} \) would determine a linear subspace of dimension \( n - 2 \) that belonged to \( \text{Sing}(Q_t) \). Since \( n \geq 4 \), this would again violate the fact that \( \dim \text{Sing}(Q_t) \leq 1 \). Thus, \( Q_t \) must be irreducible over \( \mathbb{Q} \).

We can now apply Theorem 2 of [14]. This tells us that for all \( \varepsilon > 0 \):
\[
N^*(t; x) := \# \{m \in \mathbb{Z}^n : Q_t(m) = 0; \gcd(m_1, \ldots, m_n) = 1, \|m\| \leq x\} \ll_{\varepsilon} x^{n-2+\varepsilon} \quad \text{uniformly in } t.
\]

It follows that for any \( \varepsilon > 0 \) and \( x \geq 1 \):
\[
N(t; x) := \# \{m \in \mathbb{Z}^n : Q_t(m) = 0 \text{ and } \|m\| \leq x\}
= \sum_{1 \leq d \leq x} \sum_{m \in \mathbb{Z}^n : Q_t(m) = 0; \gcd(m_1, \ldots, m_n) = d \text{ and } \|m\| \leq x} \ll_{\varepsilon} \sum_{1 \leq d \leq x} \left(\frac{x}{d}\right)^{n-2+\varepsilon} \ll_{\varepsilon} x^{n-2+\varepsilon} \sum_{1 \leq d \leq x} \frac{1}{d^{1+\varepsilon}} \ll_{\varepsilon} x^{n-2+\varepsilon} \quad \text{uniformly in } t.
\]

We deduce that for any \( \varepsilon > 0 \), we have uniformly in \( \theta \in [0, 2\pi) \setminus \{\pi/2, 3\pi/2\} \) and in \( x \geq 1 \):
\[
N_\theta(x) = \# \{(m_1, m_2) \in \mathcal{F}'(x)^2 : \cos^2(\theta(m_1, m_2)) = \cos^2(\theta)\}
= \sum_{m_2 \in \mathcal{F}'(x)} N(m_2, x) \ll_{\varepsilon} \sum_{m_2 \in \mathcal{F}'(x)} x^{n-2+\varepsilon/2} \ll_{\varepsilon} x^{n-2+\varepsilon/2} \sum_{m_2 \in \mathcal{F}'(x)} 1 \ll_{\varepsilon} x^{n-2+\varepsilon/2} x^{\varepsilon} + x^{\varepsilon/2} \ll_{\varepsilon} x^{n-1+\varepsilon}.
\]

This completes the proof of Lemma 5.

\[\square\]

Proof of Theorem 2. As in §3.1, we use the notation \( \{0 < \lambda_1 < \lambda_2 < \cdots\} \) to denote the set of distinct values of the function \( m \in \mathcal{F}' \rightarrow \|m\| \). We also define the function of \( k = (k_1, k_2) \):
\[
f_2(k) := \sum_{(m_1, m_2) \in \mathcal{F}'(x) : \|m_1\|, \|m_2\| = (\lambda_{k_1}, \lambda_{k_2})} \left\langle \frac{m_1}{\|m_1\|}, \frac{m_2}{\|m_2\|} \right\rangle^2,
\]
which implies that
\[
\zeta_{\text{ang}}(\mathcal{F}, s) = \sum_k \frac{f_2(k)}{\lambda_{k_1}^{\sigma_1} \lambda_{k_2}^{\sigma_2}} \quad \text{when } \sigma_1, \sigma_2 > e_{\mathcal{F}}.
\]

It follows from Lemma 4, Theorem 8, and (2.42) that for any \( \varepsilon > 0 \),
\[
A(f_2, x) \gg_{\varepsilon} x^{2e_{\mathcal{F}} - \varepsilon} \quad \text{when } x \to \infty.
\]
We now find a simple upper bound for $A$. For any $x$, we set

$$\Theta(x) = \{\theta(m_1, m_2) : m_1, m_2 \in \mathcal{F}(x) \} \setminus \{\pi/2, 3\pi/2\} \subset [0, 2\pi);$$

and

$$T(x) = \# \Theta(x);$$

and denote the distinct elements of $\Theta(x)$ as $\{\theta_1, \ldots, \theta_{T(x)}\}$.

It is easy to see that for any $x > 0$,

$$A(f_2, x) = \sum_{(m_1, m_2) \in \mathcal{F}(x)^2} \cos^2(\theta(m_1, m_2)) = \sum_{j=1}^{T(x)} \cos^2(\theta_j)N_{\theta_j}(x).$$

An application of Cauchy–Schwartz and the uniform estimate given by Lemma 5 then tells us

$$A(f_2, x) \leq \|\cos^2(\theta_1), \ldots, \cos^2(\theta_{T(x)})\| \cdot \|(N_{\theta_1}(x), \ldots, N_{\theta_{T(x)}}(x))\|$$

$$\leq \sqrt{T(x)} \|(N_{\theta_1}(x), \ldots, N_{\theta_{T(x)}}(x))\|$$

$$\leq x^{2\varepsilon_1 + n - 2 + \varepsilon_1} \cdot T(x).$$

Since $\omega(Z) = 2e_\mathcal{F}$ when $Z = \zeta_{\ang}^*(\mathcal{F}, s)$, it follows from (3.20) that for any $\varepsilon_0, \varepsilon_1 > 0$,

$$x^{2e_\mathcal{F} - \varepsilon_0} \ll x^{e_\mathcal{F} + n - 2 + \varepsilon_1} \cdot T(x).$$

From this, we see immediately, that if $e_\mathcal{F} > n - 2$, then for any $\varepsilon > 0$

$$\ang(x) \gg x^{e_\mathcal{F} - n + 2 + \varepsilon} \gg x [\# \mathcal{F}(x)]^{1 - (n - 2)/e_\mathcal{F} - \varepsilon}$$

uniformly in $x \gg 1$.

This completes the proof of Theorem 2.

3.3. Rooted configurations

Here, we prove the lower bound assertion (2.11) in Theorem 3.

Since we insist upon working with distinct vectors, we must first specify a suitable subset of $\mathcal{F}^{k+1}$ in order to define our ‘rooted configuration’ zeta function. Set

$$\mathcal{F}_{k+1} := (\mathcal{F})^{k+1}, \quad \mathcal{P}_{2,k+1} = \{\ell = \{u, k + 1\} : 1 \leq u \leq k\},$$

and for each $\ell = \{u < k + 1\}$

$$\mathcal{F}_\ell = \{m = (m_1, \ldots, m_{k+1}) \in \mathcal{F}_{k+1} : m_i = m_{k+1}\},$$

$$\mathcal{F}_{(k+1)}^\ell = \mathcal{F}_{k+1} \setminus \left( \bigcup_{\ell \in \mathcal{P}_{2,k+1}} \mathcal{F}_\ell \right),$$

$$\Delta_\ell(m) = \|m_u - m_{k+1}\|^2.$$

We then define (a priori formally) the rooted configuration zeta function as a function of $s = (s_1, \ldots, s_{k+1})$:

$$\zeta_{(k+1)}(s) = \sum_{\ell \in \mathcal{P}_{2,k+1}} \zeta_\ell(s), \quad \text{where} \quad \zeta_\ell(s) := \sum_{m \in \mathcal{F}_{(k+1)}^\ell} \frac{\Delta_\ell(m)}{\prod_{t=1}^{k+1} \|m_t\|^{s_\ell}}. \quad (3.22)$$

It is not difficult to see that $\zeta_{(k+1)}(s)$ converges absolutely in the product of half planes

$$\mathcal{D}_\mathcal{F} := \{s \in \mathbb{C}^{k+1} : \sigma_\ell > e_\mathcal{F} + 2 \quad \forall \ell\}.$$

In $\mathcal{D}_\mathcal{F}$, we can therefore write $\zeta_{(k+1)}(s)$ as an absolutely convergent multivariable Dirichlet series. First, we define the set of $k + 1$-tuples

$$\{\|m_1\|, \ldots, \|m_{k+1}\| \} \in \mathcal{F}_{(k+1)}^\ell \} := \{\lambda_{t_1}, \ldots, \lambda_{t_{k+1}}\},$$

where
and let $\Lambda_\ell$ denote the element of the set, defined on the right-hand side, with index vector $\ell = (\ell_1, \ldots, \ell_{k+1})$. In $D_\mathcal{F}$, then we have an identity between absolutely convergent series:

$$\zeta_{(k+1)}(s) = \sum_{\ell} \frac{f_3(\ell)}{\prod_{j=1}^{k+1} \lambda_{\ell_j}^{s_j}} = Z(f_3, s),$$

where

$$f_3(\ell) = \sum_{\{m \in \mathcal{F}_{(k+1)}; \|m\|_\ell = \Lambda_\ell\}} \left( \sum_{i \in \mathcal{P}_{2,k+1}} \Delta_i(m) \right).$$

Note: In the following, and in order to be consistent with the discussion in § 2.3, we will use the notation $Z(f_3, s)$ when referring to the series.

For $x = (x_1, \ldots, x_{k+1}) \in (0, \infty)^{k+1}$, we now define

$$\mathcal{F}_{k+1}(x) = \{ m \in \mathcal{F}_{k+1} : \|m_\ell\|_\ell \leq x_\ell \forall \ell \}; \quad \mathcal{F}_{(k+1)}(x) = \mathcal{F}_{(k+1)}(x) \cap \mathcal{F}_{k+1}(x):$$

$$A(f_3, x) = \sum_{\ell, \lambda_{\ell_j} \leq x_{\ell_j} \forall j} f_3(\ell) = \sum_{m \in \mathcal{F}_{(k+1)}(x)} \left( \sum_{i \in \mathcal{P}_{2,k+1}} \Delta_i(m) \right); \quad (3.23)$$

$$N(f_3, x) = \sum_{\ell, \lambda_{\ell_j} \leq x_{\ell_j} \forall j} f_3(\ell) \cdot \prod_{j} \left( 1 - \frac{\lambda_{\ell_j}}{x_{\ell_j}} \right).$$

It is also an elementary exercise to connect $A(f_3, x)$ to rooted configurations. For any $x$, define

$$\mathcal{C}^\lambda(x) := \{ t = (t_1, \ldots, t_k) : t_i = \|m_i - m_{k+1}\| \forall i \leq k$$

for some $(m_1, \ldots, m_k) \in \mathcal{F}_{(k+1)}(x)$; $\forall t \in \mathcal{C}^\lambda(x) \mathcal{N}_t(x) := \#(\mathcal{F}_{(k+1)}(x) \cap \{(m_1, \ldots, m_k) \in R^{kn} : \|m_j - m_{k+1}\| = t_j \forall j\});$ $\mathcal{M}^\lambda(x) := \#\mathcal{C}^\lambda(x).$

Let $t_1, \ldots, t_{\mathcal{M}^\lambda(x)}$ denote the distinct elements of $\mathcal{C}^\lambda(x)$. Writing the components of a given $t = (t_{1,j}, \ldots, t_{k,j}) \in \mathcal{C}^\lambda(x)$, we define

$$|t_j|^2 := \sum_{i=1}^{k} t_{i,j}^2.$$

Applying Cauchy–Schwarz and the definition of the summands of $A(f_3, x)$ (see (3.23)), we have the following analog of (3.4):

$$A(f_3, x) = \sum_{t \in \mathcal{C}^\lambda(x)} \mathcal{N}_t(x) \cdot |t|^2$$

$$\leq \|(|t_1|^2, \ldots, |t_{\mathcal{M}^\lambda(x)}|^2)\| \cdot \|\mathcal{N}_t(x), \ldots, \mathcal{N}_{t_{\mathcal{M}^\lambda(x)}}(x)\||. \quad (3.24)$$

We now need to recall the basic analytic properties of $Z(f_3, s)$ that are proved in [7] (see Theorem 2, p. 20ff.) in a more general setting than that needed here. The proofs of the following are very similar and can be left to the reader as simple verifications that use the decomposition (3.22) and the properties satisfied by $\zeta_2(s_1, s_2)$ (see the proof of Theorem 1).

1. The zeta function $Z(f_3, s)$ has a meromorphic extension to $C^{k+1}$ with moderate growth in $\mathcal{V} C^{k+1}$.
2. Setting $e_1, \ldots, e_{k+1}$ to denote the unit basis vectors, define the points

$$v_u = e_\mathcal{F} \cdot \sum_{r} e_r + 2e_u \quad 1 \leq u \leq k + 1.$$
These points are the vertices of the polyhedron $\Gamma(Z)$ defined in §2.3, when $Z = Z(f_3, s)$. Thus,
\[ \omega(Z) = e_x(k + 1) + 2. \]
Its unique $k$-dimensional compact face is a subset of $\sum_u \sigma_u = e_x \cdot (k + 1) + 2$. Above this polyhedron, $Z(f_3, s)$ is absolutely convergent and below it, the meromorphic extension of $Z(f_3, s)$ is not analytic.

(3) For each $1 \leq u \leq k + 1$, the $k + 1$-fold iterated residue $\text{Reg} Z(f_3, v_u) \neq 0$ and equals a sum of products of $k + 1$ standard 1-variable Cauchy residues
\[ \text{Reg} Z(f_3, v_u) = \prod_{\ell} \text{Res}_{s_\ell = v_{\ell, u}} J_{\ell, u}(s_\ell) \quad (v_u = (v_{1,u}, \ldots, v_{k+1,u})), \]
where each $J_{\ell, u}$ is analytic if $\sigma_\ell > v_{\ell, u}$ and meromorphic in the $s_\ell$ plane.

It will also be useful to be more explicit about the description in (3). Using the decomposition (3.22), and setting $s_\ell = (s_u, s_{k+1})$ when $\ell = \{u, k + 1\}$, it follows that there exists a function $E_\ell$ that is analytic in some neighborhood of each point $v_u$ such that
\[ \zeta_\ell(s) = \zeta_2(s_\ell) \cdot \prod_{j \notin \ell} \zeta(F, s_j) + E_\ell. \]
Setting $E = \sum_\ell E_\ell$, we conclude not only that
\[ Z(f_3, s) = \sum_{\ell \in P_{f, k+1}} \left( \zeta_2(s_\ell) \cdot \prod_{j \notin \ell} \zeta(F, s_j) \right) + E \]
but also that the iterated residue at $v_u$ of the function on the left equals the sum (over $\ell$) of iterated residues at $v_u$ of the sum of functions on the right that appear to the left of the plus sign. In fact, if $u \leq k$, then the discussion in [7] (see proof of Theorem 2, p. 26ff.) shows that there exists a unique term on the right whose iterated residue at $v_u$ is non-zero. This term is precisely
\[ \zeta_{\{u,k+1\}}(s_u, s_{k+1}) \cdot \prod_{j \notin \{u,k+1\}} \zeta(F, s_j). \quad (3.25) \]
It follows that the expression (2.28) is particularly simple at each $v_u$ because the value of $B$ equals 1.

As a result, the factors $J_{\ell, u}(s_\ell)$ are as follows:
\[ J_{\ell, u} = \begin{cases} 
\zeta(F, s_\ell) & \text{if } \ell \notin \{u, k + 1\}, \\
\zeta(F, s_u - 2) & \text{if } \ell = u \neq k + 1,
\end{cases} \quad \sum_{\ell \leq k} \zeta(F, s_{k+1} - 2) & \text{if } \ell = u = k + 1. \quad (3.26) \]

We now have enough information to conclude the proof of Theorem 3. The properties stated above tell us that we are justified in applying the conclusion of Theorem 8. Thus, we have the following lower bound:
\[ x^{\omega(Z) - \varepsilon_1} = x^{e_x(k + 1) + 2 - \varepsilon_1} \ll \varepsilon_1 \quad \mathcal{H}(f_3, x) \leq \mathcal{A}(f_3, x) \text{ as } x \to \infty. \quad (3.27) \]
It is a simple matter to find an upper bound for each $|t_j|^2$. For each $j = 1, \ldots, M^y(x)$:
\[ |t_j|^2 \leq kx^2 \ll x^2. \quad (3.28) \]
This, so far, has not used anything about where the points of $F$ lie.

We next apply the hypothesis that $F \subset \mathbb{Z}^n$. The same reasoning from §3.1 extends easily to treat the rooted configurations and allows us to estimate each $N_{t_j}$ uniformly in $x$ as follows.
We first choose any \( m_{k+1} \in \mathcal{F}(x) \), of which there are \( \ll \varepsilon \) possible choices. For each such choice, say \( \xi \), the other \((m_1, \ldots, m_k)\) so that
\[
(m_1, \ldots, m_k, \xi) \in \mathcal{F}(k+1) \cap \{ (m_1, \ldots, m_k) \in \mathbb{R}^{kn} : \| m_i - \xi \| = t_j \ \forall j \}
\]
evidently belong to the product of \( k \) spheres
\[
\prod_{i=1}^{k} \{ \| m_i - \xi \|^2 = t_{i,j}^2 \}.
\]
Thus, the needed estimate for any \( N_{\varepsilon_j}(x) \) is
\[
\ll \varepsilon_1, \theta_1, \theta_2 \prod_{i=1}^{k} t_{i,j}^{-2 + \theta_2} \cdot t_{k+1,j}^{-\theta_1} \ll x^{(e_x + \theta_1 + k(n-2 + \theta_2))}.
\]
Moreover, this bound is uniform in \( t \).

Combining the lower bound (3.27) with (3.24) and the preceding estimates now gives us the following analog of (3.10) for the number of distinct rooted configurations:
\[
x^{e_x(k+1) + 2 - \varepsilon_1} \ll \varepsilon_1, (f_3, x) \ll \varepsilon_1, x^{(e_x + \theta_1 + k(n-2 + \theta_2))^2} M^\vee(x).
\]
The exponent of \( x \) that determines the lower bound for \( M^\vee(x) \) is therefore of interest only when it is positive, that is, we should choose \( \varepsilon_1, \theta_1, \theta_2 \) so that
\[
[e_x(k + 1) + 2 - \varepsilon_1] - [e_x + \theta_1 + k(n-2 + \theta_2) + 2] > 0.
\]
As a result, if \( e_x > n - 2 \), then for any \( \varepsilon_2 > 0 \) so that \( e_x - n + 2 > \varepsilon_2 \) (that is, for 'sufficiently small' \( \varepsilon \)), we can choose \( \varepsilon_1 \) and \( \theta_1, \theta_2 > 0 \) so that the inequality above is satisfied; and when this occurs, we have the following analogs of (3.12) and (3.14) when \( x \to \infty \):
\[
M^\vee(x) \gg \varepsilon_2 x^{k[e_x-n+2]-\varepsilon_2} \quad \text{and} \quad M^\vee(x) \gg \varepsilon_2, \varepsilon_3 \#(\mathcal{F}(x))^{(k[e_x-n+2]-\varepsilon_2)/(e_x+\varepsilon_3)}.
\]
We now choose for any \( \varepsilon > 0 \) values for \( \varepsilon_2, \varepsilon_3 \) so that the exponent of \( \# \mathcal{F}(x) \) is larger than \( k(1 - (n-2)/e_x) - \varepsilon \), and conclude the proof of Theorem 3, having now shown the asymptotic lower bound:
\[
\text{Root}_{k, \mathcal{F}}(x) \gg \varepsilon \#(\mathcal{F}(x))^{k[1-(n-2)/e_x]-\varepsilon} \quad \text{as} \ x \to \infty.
\]

4. Simplex zeta functions

In order to estimate from below the number of distinct volumes determined by \( n \)-simplices of a compatible discrete self-similar set, we introduce a 'simplex zeta function', whose basic properties are proved in Theorem 9 (§4.2). This function, associated to \( \mathcal{F} \), is defined formally as a function of \( s = (s_1, \ldots, s_{n+1}) \) as follows:
\[
\zeta_{\text{sim}}(\mathcal{F}; s) := \sum_{m_1, \ldots, m_{n+1} \in \mathcal{F}^s} \frac{\det^2(m_1 - m_{n+1}, \ldots, m_n - m_{n+1})}{\| m_1 \|^s_1 \cdots \| m_{n+1} \|^s_{n+1}}.
\]
We also define (formally) the determinant zeta function associated to \( \mathcal{F} \) as a function of \( s = (s_1, \ldots, s_n) \) as follows:
\[
\zeta_{\text{det}}(\mathcal{F}; s) := \sum_{m_1, \ldots, m_n \in \mathcal{F}^s} \frac{\det^2(m_1, \ldots, m_n)}{\| m_1 \|^s_1 \cdots \| m_n \|^s_n}.
\]
The first two subsections present the basic analytic properties of these Dirichlet series that are needed in §5 to prove Theorem 4.
4.1. Meromorphic continuation of $\zeta_{\text{det}}(F; s)$ and definition of thick self-similar sets

Applying standard formulae and properties for the determinant, and using the method of analytic continuation from [6] for any compatible self-similar set $F$, we easily prove the following result (recall that $D_F = e_F + 2$).

**Proposition 1.** The determinant zeta function $\zeta_{\text{det}}(F, s)$ of $F$ converges absolutely in the domain $\bigcap_{i=1}^n \{ \sigma_i > D_F \}$, and has a meromorphic continuation with moderate growth to $\mathbb{C}^n$. In addition, its polar locus is a subset of

$$\mathcal{P}_{\text{det}}(F) := \bigcup_{\alpha \in \mathbb{N}_0^n} \bigcup_{k \in \mathbb{N}_0^n} \left\{ s_i + 2 - k_i : \sum_{j=1}^r \lambda_j^\alpha c_j^{-s_i} = 1 \right\}. \tag{4.3}$$

Moreover, the function

$$s \mapsto \tilde{\zeta}_{\text{det}}(F; s) := \left[ \prod_{j=1}^n \prod_{\alpha \in \mathbb{N}_0^n : |\alpha| = 2} \left( \sum_{j=1}^r \lambda_j^\alpha c_j^{-s_i} - 1 \right) \right] \zeta_{\text{det}}(F; s)$$

has a holomorphic continuation to the set $\bigcup_{i=1}^n \{ \sigma_i > D_F - 1 \}$ where it satisfies the estimate

$$\tilde{\zeta}_{\text{det}}(F; \sigma + i\tau) \ll_n \prod_{i=1}^n (1 + |\tau_i|).$$

**Definition 4.** The set $F$ is thick if the point $(D_F, \ldots, D_F)$ is a pole of $\zeta_{\text{det}}(F; s)$ and the iterated residue

$$\text{Res}_{s_i = D_F} \cdots \text{Res}_{s_n = D_F} (\zeta_{\text{det}}(F; s)) \neq 0.$$

**Proof of Proposition 1.** It follows from Hadamard’s inequality (that is, $|\det(m_1, \ldots, m_n)| \leq \|m_1\| \cdots \|m_n\|$) that $\zeta_{\text{det}}(F; s)$ converges absolutely on the set $\bigcup_{i=1}^n \{ \sigma_i > D_F \}$. It is then straightforward to verify that if $\sigma_i > D_F \forall i$, then

$$\zeta_{\text{det}}(F; s) = \sum_{\omega_1, \omega_2 \in S_n} \text{sgn} (\omega_1 \omega_2) \prod_{i=1}^n \zeta_F(e_{\omega_1(i)} + e_{\omega_2(i)}; s_i). \tag{4.4}$$

Proposition 1 is an immediate consequence of these two properties and Lemma 2 in §2.2.2.

4.2. Properties of $\zeta_{\text{sim}}(F; s)$ when $F$ is thick

Elementary properties of the determinant and Lemma 2 tell us that the simplex zeta function $\zeta_{\text{sim}}(F; s)$ (see (4.1)) has a meromorphic extension to $\mathbb{C}^{n+1}$ outside a domain of absolute convergence determined by $D_F$. As a result, there is a polyhedron $\Gamma_{\text{sim}}(F) \subset \mathbb{R}^{n+1}$ that determines the real part of its boundary of analyticity. Of particular interest here is the fact that when $F$ is thick (see Definition 4), the vertices of $\Gamma_{\text{sim}}(F)$ can be explicitly determined. This is the point of our first result.

**Theorem 9.** Let $F$ be a compatible self-similar set. Then $\zeta_{\text{sim}}(F; s)$ satisfies the following four properties:

(i) the formal series defining $\zeta_{\text{ang}}(F, s)$ converges absolutely in the domain $\bigcap_{i=1}^{n+1} \{ \sigma_i > D_F \}$ and has a meromorphic extension to $\mathbb{C}^{n+1}$ with moderate growth;

(ii) the zeta function $\zeta_{\text{ang}}(s)$, defined by this series, satisfies the Hypotheses (I), (II), and either (III) or (IV), which are needed to apply Theorem 7;
(iii) assume that $\mathcal{F}$ is thick. For each $u = 1, \ldots, n + 1$, define the point

$$v_u = (D_{\mathcal{F}}, D_{\mathcal{F}}, \ldots, D_{\mathcal{F}}) - 2e_u \in \mathbb{R}^{n+1} := (v_{1,u}, \ldots, v_{n+1,u}).$$

Then:

(a) the vertex set (see (2.29)) is $\mathcal{V}(\zeta_{\text{sim}}) = \{v_u \mid u = 1, \ldots, n + 1\}$;

(b) for each $u$, the iterated residue

$$\text{Reg} \zeta_{\text{sim}}(v_u) = \text{Res}_{s_1 = v_{1,u}} \cdots \text{Res}_{s_{n+1} = v_{n+1,u}} (\zeta_{\text{sim}}) \neq 0.$$

Proof of Theorem 9. First, we remark that

$$2 \det (m_1 - m_{n+1}, \ldots, m_n - m_{n+1})$$

$$= \left( \det (m_1, \ldots, m_n) - \sum_{j=1}^n \det (m_1, \ldots, m_{j-1}, m_{n+1}, m_j, \ldots, m_n) \right)^2$$

$$= \det (m_1, \ldots, m_n) + \sum_{j=1}^n 2 \det (m_1, \ldots, m_{j-1}, m_{n+1}, m_j, \ldots, m_n)$$

$$+ 2H(m_1, \ldots, m_{n+1}), \quad (4.5)$$

where

$$H(m_1, \ldots, m_{n+1})$$

$$= \sum_{1 \leq j < j' \leq n} \det (m_1, \ldots, m_{j-1}, m_{n+1}, m_j, \ldots, m_{n+1}) \times \det (m_1, \ldots, m_{j'-1}, m_{n+1}, m_{j'+1}, \ldots, m_n)$$

$$- \det (m_1, \ldots, m_n) \sum_{j=1}^n \det (m_1, \ldots, m_{j-1}, m_{n+1}, m_j, \ldots, m_n).$$

By using the determinant expansion, it is easily seen that $H(m_1, \ldots, m_{n+1})$ is a finite linear combination of term of the form $(\prod_{1 \leq i \leq n+1, i \neq a, b} m_i^{\alpha_i} m_a^{\alpha_a} m_b^{\alpha_b})$, where $a \neq b \in \{1, \ldots, n + 1\}$ and for each $i$, $\alpha_i \in \mathbb{N}_0$ satisfies $|\alpha_i| = 2$.

It follows from this that

$$\zeta_{\text{sim}}(\mathcal{F}; \hat{s}) = \sum_{i=1}^{n+1} \zeta_{\det}(\mathcal{F}; s_1, \ldots, s_i, \ldots, s_{n+1}) \zeta(\mathcal{F}, s_i) + R(\mathcal{F}; \hat{s}), \quad (4.6)$$

where $R(\mathcal{F}; \hat{s})$ is a linear combination of terms of the form

$$\left( \prod_{1 \leq k \leq n+1 \atop k \neq a, b} \zeta_{\mathcal{F}}(\alpha_k; s_k) \right) \zeta_{\mathcal{F}}(e_{i_a}; s_a) \zeta_{\mathcal{F}}(e_{i_b}; s_b) \quad (|\alpha_k| = 2 \forall k).$$

In other words, $R$ is a linear combination of products of Dirichlet series taken over all pairs $a \neq b$ and $k \neq a, b$ such that the $k^{th}$ factor is a zeta function

$$\zeta_{\mathcal{F}}(m_{\ell_1,k}^{\alpha_{\ell_1,k}} m_{\ell_2,k}^{\alpha_{\ell_2,k}}, s_k)$$

such that $\alpha_{\ell_1,k} + \alpha_{\ell_2,k} = 2$,

while the product over $a \neq b$ equals

$$\zeta_{\mathcal{F}}(m_{i_a,a}; s_a) \zeta_{\mathcal{F}}(m_{i_b,b}; s_b).$$

As a result, the largest real part of any pole of the $k^{th}$ factor is at most $\sigma_k = D_{\mathcal{F}}$ $(k \neq a, b)$, but the largest real part of the two other factors is at most $\sigma_a, \sigma_b = D_{\mathcal{F}} - 1$. 


Thus, it follows from Lemma 2 that \( \zeta_{\text{sim}}(F; \dot{s}) \) converges absolutely in the domain \( \bigcap_{i=1}^{n+1} \{ \sigma_i > D_F \} \), and has a meromorphic extension to \( \mathbb{C}^{n+1} \) with moderate growth along \( i \mathbb{R}^{n+1} \) (outside the set of points within a given positive distance to a non-real pole) whenever \( \sigma \) is confined to a compact set. It follows also that \( \Gamma(\zeta_{\text{sim}}) = \text{convex hull}\{ u: u = 1, \ldots, n+1 \} \).

This finishes the proof of Parts (i), (ii), and (iii)a.

Part (iii)b of the theorem follows by combining (4.6) with the properties of \( \zeta_F(\alpha, s) \), proved in Lemma 2, and the thickness of \( F \). In particular, each \( v_u \) is a pole of \( \zeta_{\text{sim}}(F; \dot{s}) \) at which the \( n + 1 \)-fold iterated residue is non-zero, and no pole of \( R \) can belong to the set \( \{ v_u \}_u \). Moreover, it is not difficult to verify (details left to the reader) that the real part of any pole of \( R \) must lie either in the interior of a face of the polyhedron \( \Gamma(\zeta_{\text{sim}}) \) or below it. This completes the proof of Theorem 9.

4.3. Three examples of thick \( F \)

It is not difficult to find some simple examples of thick compatible self-similar sets.

**Example 1.** Symmetric \( F \) (that is, \( \forall \tau \in S_n, \ (m_{\tau(1)}, \ldots, m_{\tau(n)}) \in F \) if and only if \( (m_1, \ldots, m_n) \in F \)).

**Claim 1.** If \( F \) is symmetric and if there exists \( j_0 \in \{1, \ldots, r\} \) and \( i_1 \neq i_2 \in \{1, \ldots, n\} \) such that \( \lambda_{j_0, i_1} \lambda_{j_0, i_2} \neq 1 \), then \( F \) is thick.

**Proof.** Let \( \alpha \in \mathbb{N}_0^n \) be such that \( |\alpha| = 2 \) (that is, \( i, j \in \{1, \ldots, n\} \) exist so that \( \alpha = e_i + e_j \)). The symmetry of \( F \) implies that

\[
\zeta_F(\alpha; s) = \frac{1}{n} \zeta_F(s - 2) \quad \text{if } i = j,
\]

\[
\zeta_F(e_i + e_j; s) \quad \text{if } i \neq j.
\]

Lemma 2 and (4.4) then imply that the function

\[
V(s_1, \ldots, s_n) := \prod_{i=1}^{n} (s_i - D_F) \zeta_{\text{det}}(F; s_1, \ldots, s_n)
\]

\[
= \sum_{\omega_1, \omega_2 \in S_n} \text{sgn}(\omega_1 \omega_2) \prod_{i=1}^{n} (s_i - D_F) \zeta_F(e_{\omega_1(i)} + e_{\omega_2(i)}; s_i)
\]

is holomorphic at \( (D_F, \ldots, D_F) \) and satisfies

\[
V(D_F, \ldots, D_F) = \sum_{\omega_1, \omega_2 \in S_n} \text{sgn}(\omega_1 \omega_2) \prod_{i=1}^{n} \text{Res}_{s=D_F} \zeta_F(e_{\omega_1(i)} + e_{\omega_2(i)}; s)
\]

\[
= n! \left( \frac{1}{n} \text{Res}_{s=D_F} \zeta_F(s - 2) \right)^n = \frac{n!}{n^n} (\text{Res}_{s=e_F} \zeta_F(s))^n > 0.
\]

This completes the proof of Claim 1.

**Example 2.** Let \( p \) be a prime number. Pascal’s triangle mod \( p \) is defined by

\[
\text{Pas}(p) = \left\{ (m_1, m_2) \in \mathbb{N}_0^2: m_1 \geq m_2 \text{ and } \binom{m_1}{m_2} \equiv 0 \ (\text{mod } p) \right\}.
\]
A basic result of [5] (see Theorem 1) tells us that \( \Pas(p) \) is a compatible self-similar set with upper Minkowski dimension
\[
\epsilon_p = \ln(p(p + 1)/2) / \ln p,
\]
where, for simplicity, we write \( \epsilon_p \) rather than \( \epsilon_{\Pas(p)} \). Similarly, we write \( D_p \) in place of \( D_{\Pas(p)} \).

(Not that the set is compatible since the orthogonal transformation for each similarity is the identity on \( \mathbb{R}^2 \).)

**Claim 2.** The set \( \Pas(p) \) is thick.

**Proof.** Lemma 2 implies that the function
\[
V(s_1, s_2) := (s_1 - D_p) (s_2 - D_p) \det(\Pas(p); s_1, s_2)
\]
is holomorphic at \( (D_p, D_p) \). In addition, we know from §3.1 in [7] that \( \det(\Pas(p); s, s) \) has a pole of order 2 at \( s = D_p \). It follows that \( V(D_p, D_p) \neq 0 \). This completes the proof of Claim 2.

**Example 3.** The construction of Pascal pyramids mod \( p \) is a natural generalization of Pascal triangles that uses multinomial coefficients. For any integral vector \( m = (m_1, \ldots, m_n) \in \mathbb{N}_0^n \), the multinomial coefficient determined by \( m \) is the integer
\[
\text{Multnom}_n(m) := \frac{(m_1 + \cdots + m_n)!}{m_1! \cdots m_n!}.
\]

For any prime number \( p \), the analog of Pascal’s triangle mod \( p \) is called the Pascal pyramid mod \( p \) and is defined as the set
\[
\mathcal{M}_{n,p} := \{ m \in \mathbb{N}_0^n | \text{Multnom}_n(m) \equiv 0 \pmod{p} \}.
\]

Lemma 4 of [6] tells us that
\[
\mathcal{M}_{n,p} = \bigsqcup_{r \in I(p,n)} f_r(\mathcal{M}_{n,p}),
\]
where
\[
\begin{align*}
& (i) \ I(p, n) := \{ r \in \{0, \ldots, p - 1\}^n; |r| = r_1 + \cdots + r_n \leq p - 1 \}; \\
& (ii) \ f_r = p \text{Id}_{\mathbb{R}^n} + r \text{ for any } r \in I(p,n).
\end{align*}
\]

It is also useful to note that
\[
\#I(p, n) = \binom{p + n - 1}{p - 1}.
\]

It follows that \( \mathcal{M}_{n,p} \) is a compatible self-similar set of upper Minkowski dimension (see [6, Corollary 2]):
\[
\epsilon_{n,p} := \frac{\ln \left( \binom{p + n - 1}{p - 1} \right)}{\ln p}.
\]

We prove the following.

**Claim 3.** For any prime \( p \) and any \( n \geq 2 \), \( \mathcal{M}_{n,p} \) is thick.

To prove Claim 3, we need the following result that extends Lemma 2 when \( \mathcal{F} = \mathcal{M}_{n,p} \).
Lemma 6. Set \( A := \text{Res}_{s=\epsilon_n,p+2} \zeta_{\mathcal{M}_{n,p}}(2e_1; s) \) and \( B := \text{Res}_{s=\epsilon_n,p+2} \zeta_{\mathcal{M}_{n,p}}(e_1 + e_n; s) \). Then:

1. \( A = \text{Res}_{s=\epsilon_n,p+2} \zeta_{\mathcal{M}_{n,p}}(2e_1; s) \) for all \( i = 1, \ldots, n \);
2. \( B = \text{Res}_{s=\epsilon_n,p+2} \zeta_{\mathcal{M}_{n,p}}(e_1 + e_j; s) \) for all \( i \neq j \in \{1, \ldots, n\} \);
3. \( A > B \).

Proof of Lemma 6. Since \( \mathcal{M}_{n,p} \) is invariant under the action of the symmetric group \( S_n \) that exchanges the \( m_i \), it is clear that Parts (i) and (ii) follow immediately. Thus, only Part (iii) requires justification.

Set \( H(s) := \zeta_{\mathcal{M}_{n,p}}(2e_1; s) - \zeta_{\mathcal{M}_{n,p}}(e_1 + e_n; s) \). It follows from Lemma 2 that \( H(s) \) converges absolutely in the half plane \( \{ \sigma > \epsilon_n,p + 2 \} \) and has a meromorphic continuation to the entire \( s \)-plane. Moreover, since each eigenvalue of the similarities \( f_r \) equals 1, it follows that \( s = \epsilon_n,p + 2 \) is at most a simple pole of \( H(s) \) with residue

\[
\text{Res}_{s=\epsilon_n,p+2} H(s) = A - B. \tag{4.9}
\]

The symmetry of \( \mathcal{M}_{n,p} \) implies that for \( \sigma > \epsilon_n,p + 2 \) we have

\[
2H(s) = \zeta_{\mathcal{M}_{n,p}}(2e_1; s) + \zeta_{\mathcal{M}_{n,p}}(2e_n; s) - 2\zeta_{\mathcal{M}_{n,p}}(e_1 + e_n; s)
\]

\[
= \sum_{m \in \mathcal{M}_{n,p}} \frac{(m_1 - m_n)^2}{||m||^s}.
\tag{4.10}
\]

This implies that \( A \geq B \). So to conclude, it suffices to prove that \( s = \epsilon_n,p + 2 \) is indeed a pole of \( H(s) \).

To show this, we first note that

\[
2H(s) = \sum_{m \in \mathcal{M}_{n,p}} \frac{(m_1 - m_n)^2}{||m||^s}
\]

\[
\equiv \sum_{r \in I(p,n)} \sum_{m \in \mathcal{M}_{n,p}} \frac{p^2(m_1 - m_n)^2 + 2(r_1 - r_n)(m_1 - m_n) + (r_1 - r_n)^2}{||pm + r||^s}.
\]

This implies (see [6], proofs of Lemma 1, (29)–(30), and Corollary 2) that if \( \sigma > \epsilon_n,p + 2 \), then

\[
2(1 - I(p,n)p^{2-\sigma})H(s) = \sum_{r \in I(p,n)} \sum_{m \in \mathcal{M}_{n,p}} \frac{2(r_1 - r_n)(m_1 - m_n) + (r_1 - r_n)^2}{||pm + r||^s}
\]

\[
= O_{\sigma} \left( \sum_{m \in \mathcal{M}_{n,p}} \frac{1}{||m||^{\sigma - 1}} \right) = O_{\sigma}(\zeta(\mathcal{M}_{n,p}; \sigma - 1)).
\]

Therefore, \( s \mapsto 2(1 - I(p,n)p^{2-\sigma})H(s) \) has a holomorphic extension to \( \{ \sigma > \epsilon_n,p + 1 \} \).

Setting \( \sigma_a \) to denote the abscissa of convergence of \( 2H(s) \), a Dirichlet series with non-negative coefficients, we note that a classical result of Landau implies that \( \sigma_a \) must be a singular point of \( 2H(s) \). Thus, since \( s = \epsilon_n,p + 2 \) is the only real root of the equation \( 1 - I(p,n)p^{2-\sigma} = 0 \), we deduce that

\[
\sigma_a = \epsilon_n,p + 2 \quad \text{or} \quad \sigma_a \leq \epsilon_n,p + 1. \tag{4.11}
\]

By using the fact that

\[
[m = (m_1, \ldots, m_n) \in \mathcal{M}_{n,p} \text{ and } m_n = 0] \quad \text{if and only if} \quad m' = (m_1, \ldots, m_{n-1}) \in \mathcal{M}_{n-1,p}.
\]
we deduce that

\[ 2H(\sigma) = \sum_{m \in \mathcal{M}_{n,p}} \frac{(m_1 - m_n)^2}{\|m\|^\sigma} \geq \sum_{m \in \mathcal{M}_{n,p}^{\prime}} \frac{(m_1 - 0)^2}{\|m\|^\sigma} \]

\[ = \sum_{m' = (m_1, \ldots, m_{n-1}) \in \mathcal{M}_{n-1,p}^{\prime}} \frac{m_1^2}{\|m'\|^\sigma}. \]

The symmetry of \( \mathcal{M}_{n-1,p} \) implies then that:

\[ 2H(\sigma) \geq \frac{1}{n-1} \sum_{i=1}^{n-1} \sum_{m' = (m_1, \ldots, m_{n-1}) \in \mathcal{M}_{n-1,p}^{\prime}} \frac{m_i^2}{\|m'\|^\sigma} = \frac{1}{n-1} \sum_{m' \in \mathcal{M}_{n-1,p}^{\prime}} \frac{1}{\|m'\|^\sigma - 2}. \]

It follows that

\[ \sigma_\alpha \geq e_{n-1,p} + 2. \tag{4.12} \]

Moreover, since \((p - 1)(n - 1) > 0\) implies that \((p + n - 1)/n < p\), we see that

\[ \binom{p + n - 1}{p - 1} = \frac{(p + n - 1)(p + n - 2) \cdots p}{n!} < p \cdot \frac{(p + n - 2)(p + n - 3) \cdots p}{(n - 1)!} = p \cdot \binom{p + n - 2}{p - 1}. \]

As a result, we have

\[ \ln \binom{p + n - 1}{p - 1} < \ln \binom{p + n - 2}{p - 1} + \ln p. \]

This implies the estimate

\[ e_{n,p} < e_{n-1,p} + 1. \]

It then follows from (4.11) and (4.12) that \( \sigma_\alpha \leq e_{n,p} + 1 \) is not possible. Thus, \( \sigma_\alpha = e_{n,p} + 2 \), which implies \( s = e_{n,p} + 2 \) must be a pole of \( H(s) \). This finishes the proof of the lemma.

**Finishing the proof of Claim 3**

The identity (4.4) implies that for \( \sigma > e_{n,p} + 2 \) we have

\[ \zeta_{\det}(\mathcal{M}_{n,p}; s, \ldots, s) = \sum_{\omega_1, \omega_2 \in S_n} \text{sgn}(\omega_1 \omega_2) \prod_{i=1}^{n} \zeta_{\mathcal{M}_{n,p}}(\mathbf{e}_{\omega_1(i)} + \mathbf{e}_{\omega_2(i)}, s) \]

\[ = \sum_{\omega_1, \omega_2 \in S_n} \text{sgn}(\omega_1 \omega_2^{-1}) \prod_{i=1}^{n} \zeta_{\mathcal{M}_{n,p}}(\mathbf{e}_{\omega_1(i)}^{-1} + \mathbf{e}_i, s) \]

\[ = n! \sum_{\omega \in S_n} \text{sgn}(\omega) \prod_{i=1}^{n} \zeta_{\mathcal{M}_{n,p}}(\mathbf{e}_{\omega(i)} + \mathbf{e}_i, s). \]

We deduce then from Lemmas 2 and 6 that

\[ \text{Res}_{s = e_{n,p} + 2} \cdots \text{Res}_{s = e_{n,p} + 2} (\zeta_{\det}(\mathcal{M}_{n,p}; s)) \]

\[ = \lim_{s \to e_{n,p} + 2} (s - (e_{n,p} + 2))^n \zeta_{\det}(\mathcal{M}_{n,p}; s, \ldots, s) \]

\[ = n! \sum_{\omega \in S_n} \text{sgn}(\omega) A^{\#(i; \omega(i) = i)} B^{n - \#(i; \omega(i) = i)} \]

\[ = n! \det \mathbf{M}, \]

where \( \mathbf{M} = (a_{i,j}) \) is the \( n \times n \) matrix defined by \( a_{i,j} = A \) if \( i = j \) and \( a_{i,j} = B \) otherwise.
The third point of Lemma 6 implies that $A \neq B$. Since $A, B \geq 0$ follows from the fact that the two Dirichlet series whose difference equals $H(s)$ have non-negative coefficients, we conclude that $A \neq \pm B$. An elementary determinant calculation, left to the reader, then shows that $\det M \neq 0$. This completes the proof of Claim 3.

5. **Distinct volumes of $n$-simplices when $F$ is thick**

We prove Theorem 4 from which Theorem 5 is an immediate corollary. To do this, we apply the same method as in § 3. Thickness of $F$ allows us to apply Theorem 8 that gives the needed lower bound on the averages of volumes of simplices. The upper bound uses a diophantine argument that is essentially elementary in nature.

We start with the set of distinct values of $\{\|m_j\| : m_j \in F\}$, which we denote as $\{\lambda_1 < \lambda_2 < \cdots \}$. Given any index vector $J = (j_1, \ldots, j_{n+1}) \in \mathbb{N}^{n+1}$, we set $\Lambda_J = (\lambda_{j_1}, \ldots, \lambda_{j_{n+1}})$ to denote a point in the set of vectors

$$\mathcal{L} = \{\|m_1\|, \ldots, \|m_{n+1}\| : m_i \in F \ \forall i\}.$$  

In addition, for any $\Lambda \in \mathcal{L}$ it is convenient to define

$$\mathcal{X}(\Lambda) = \{m_1, \ldots, m_{n+1} \in F^{n+1} : \|m_1\|, \ldots, \|m_{n+1}\| = \Lambda\}.$$  

We then have a representation of $\zeta_{\text{sim}}(\hat{s})$ (valid if each $\sigma_i > D_F$) as an absolutely convergent Dirichlet series

$$\zeta_{\text{sim}}(\hat{s}) = \sum_{J = (j_1, \ldots, j_{n+1})} \frac{f_4(J)}{\Lambda_J^s} := Z(f_4, \hat{s}) \left(\Lambda_J^s := \prod_{k=1}^{n+1} \lambda_{j_k}^{\sigma_k}\right), \quad (5.1)$$  

where

$$f_4(J) := \sum_{\{m_1, \ldots, m_{n+1}\} \in \mathcal{X}(\Lambda_J)} 2\det(m_1 - m_{n+1}, \ldots, m_n - m_{n+1}).$$

The polyhedron $\Gamma(Z)$, when $Z = Z(f_4, \hat{s})$, is described by Theorem 9. The vertices $e \in \Gamma(Z)$ are determined in Part 2 of this theorem. It follows that a formula for the weight of each $e$ is as follows:

$$|e| = nD_F + e_F = \omega(Z).$$

Since $F$ is thick, Theorem 8 applies and gives us an explicit and non-trivial lower bound for the averages of the $f_4(J)$:

$$\mathcal{A}(f_4, x) := \sum_{\{\Lambda \in \mathcal{L}, \Lambda_J \subseteq (x, \ldots, x)\}} f_4(J) \gg_{\theta} x^{nD_F + e_F - \theta} \text{ as } x \to \infty. \quad (5.2)$$

Three steps are needed to finish the proof.

**Step 1:** Using the above notation, we denote the set (of distinct values) as follows:

$$D(x) := \left\{\det(m_1 - m_{n+1}, \ldots, m_n - m_{n+1}) : (m_1, \ldots, m_{n+1}) \in \prod_{j} F(x)\right\},$$  

$$D(x) = \#D(x),$$

and write the distinct elements of $D(x)$ as $D(x) = \{\delta_1, \ldots, \delta_{D(x)}\}$.

For given $z \in F(x)$, we also set

$$N_\delta(z) = \# \left\{(m_1, \ldots, m_n) \in \prod_{j=1}^{n} F(x) : |\det(m_1 - z, \ldots, m_n - z)| = \delta\right\}. \quad (5.3)$$
We first note that
\[ A(f_4, x) = \sum_{\delta \in D(x)} N_\delta \cdot \delta^2, \quad \text{where } N_\delta = \sum_{z \in \mathcal{F}(x)} N_\delta(z). \]

Applying Cauchy–Schwartz tells us
\[ A(f_4, x) \leq \|\delta^{(2)}(x)\| \cdot \|N(x)\|, \]
where \( \delta^{(2)}(x) := (\delta^2_1, \ldots, \delta^2_{D(x)}), \quad N(x) := (N_{\delta_1}, \ldots, N_{\delta_{D(x)}}). \)

Using Hadamard’s inequality, we bound each component of \( \delta^{(2)}(x) \) as follows. Setting
\[ h(\delta) = \inf \{ \tau_1 \cdots \tau_n : \exists (m_1 - z, \ldots, m_n - z) \in \prod_j (\mathcal{F}(x) - \mathcal{F}(x)) \text{ such that} \]
\[ \|m_j - z\| = \tau_j \forall j \text{ and } |\det(m_1 - z, \ldots, m_n - z)| = \delta, \]
it is clear that \( |\delta| \leq h(\delta) \ll x^n \) for all \( \delta \). Thus,
\[ \|\delta^{(2)}(x)\| \ll x^{2n} \cdot \sqrt{D(x)}. \tag{5.5} \]

We deduce that
\[ A(f_4, x) \ll x^{2n} \cdot (\max_i N_{\delta_i}) \cdot D(x). \tag{5.6} \]

**Step 2:** Evidently, (5.6) reduces the problem of bounding \( A(f_4, x) \) to that of bounding the \( N_{\delta_i} \). When \( \mathcal{F} \subset \mathbb{Z}^n \), this reduces to a linear diophantine problem that can be solved in an elementary way. The important point is that the solution be uniform in a set of parameters that will be specified below.

**Note.** Throughout the discussion, we choose any \( z \in \mathcal{F}(x) \) and fix it.

We then write for any \( (m_1, \ldots, m_n) \in \prod_{1 \leq j \leq n} \mathcal{F}(x) \):
\[ (m_1 - z, m_2 - z, \ldots, m_n - z) := (m_1^*, m') = (m_{11}^*, \ldots, m_{1n}^*, m'). \]

Expanding out the equation \( |\det(m_1 - z, \ldots, m_n - z)| = \delta \) along the first row of the matrix gives two possible linear equations in the unknowns \( m_{11}^* = m_{11} - z_i \)
\[ \sum_i \Delta_i(m') \cdot m_{1i}^* = \pm \delta, \tag{5.7} \]
where \( \Delta_i(m') \) equals the product of \((-1)^{1+i} \) with the \((n - 1) \times (n - 1)\) minor with first row and \( i^{th} \) column deleted. Setting \( \Delta(m') = (\Delta_1(m'), \ldots, \Delta_n(m')) \), define the set of coefficient vectors
\[ \mathcal{Y}(\delta, z) = \left\{ \Delta(m') : m' \in \prod_{j \geq 2} (\mathcal{F}(x) - z) \right\}, \]
and for any \( \Delta \in \mathcal{Y}(\delta, z) \) define the number of solutions to (5.7):
\[ N(\delta, z, \Delta) = \# \{ m_{1i}^* \in \mathcal{F}(x) - z : m_{1i}^* \text{ is a solution of (5.7) whose coefficient vector } = \Delta \}. \]

It follows that (see (5.3))
\[ N_\delta(z) = \sum_{\Delta \in \mathcal{Y}(\delta, z)} N(\delta, z, \Delta). \tag{5.8} \]

The estimate we need for \( N_\delta(z) \) should be uniform in \( \delta \) and elements \( \Delta \) of \( \mathcal{Y}(\delta, z) \). This can be achieved by an elementary argument as follows.
Lemma 7. For all \( \mu, \nu, \gamma > 0 \), we have

\[
\# \{ (w_1, \ldots, w_n) \in \mathbb{Z}^n : \sum_{i=1}^{n} f_i w_i = \delta \text{ and } \max_i \{|w_i|\} \leq B \} \ll_{\mu, \nu, \gamma, \varepsilon} B^{n-1+\varepsilon} \tag{5.9}
\]

uniformly in

(i) \( B \in [\nu, \infty) \),
(ii) \( \delta \in [-\gamma B^\nu, \gamma B^\nu] \cap \mathbb{Z} \),
(iii) \( F = (f_1, \ldots, f_n) \in [-\gamma B^\nu, \gamma B^\nu]^n \cap (\mathbb{Z}^n - \{0\}) \).

Proof of Lemma 7. There are two parts to the proof.

Part 1: Assume \( n = 2 \). Set

\[
T(\delta, F, B) := \# \{ (w_1, w_2) \in \mathbb{Z}^2 : f_1 w_1 + f_2 w_2 = \delta \text{ and } \inf_i \{|w_i|\} \leq B \} \text{ if } f_1 f_2 \neq 0;
\]

or

\[
T(\delta, F, B) := \# \{ (w_1, w_2) \in \mathbb{Z}^2 : f_1 w_1 + f_2 w_2 = \delta \text{ and } \max_i \{|w_i|\} \leq B \} \text{ if } f_1 f_2 = 0.
\]

We prove that for any \( \mu, \nu, \gamma > 0 \), in terms of which are defined the constraints (i)–(iii), the bound

\[
T(\delta, F, B) \ll_{\mu, \nu, \gamma, \varepsilon} B^{1+\varepsilon} \tag{5.10}
\]

holds uniformly over \( \delta \) and \( F \in \mathbb{Z}^2 - \{(0,0)\} \) (satisfying (ii)–(iii)).

Proof. We first assume \( f_1 \cdot f_2 \neq 0 \). For simplicity, we also assume \( f_1, f_2 > 0 \). The reader will easily be able to modify the discussion below if at least one of the \( f_i \) is negative by an appropriate use of absolute values.

Set \( \beta = \text{gcd}(f_1, f_2) \). Evidently, if \( \beta \nmid \delta \), then \( T(\delta, F, B) = 0 \). If \( \beta \mid \delta \), then integers \( f'_1, f'_2 \), and \( \delta' \) exist such that

\[
f_1 = \beta f'_1, \quad f_2 = \beta f'_2, \quad \text{and } \delta = \beta \delta'.
\]

Thus,

\[
T(\delta, F, B) = \# \{ (w_1, w_2) \in \mathbb{Z}^2 : f'_1 w_1 + f'_2 w_2 = \delta' \text{ and } \inf_i \{|w_i|\} \leq B \},
\]

and the elements of the set whose cardinality equals \( T(\delta, F, B) \) are of the form

\[
w_1 = u_1 + kf'_2 \quad \text{and} \quad w_2 = u_2 - kf'_1 \quad (k \in \mathbb{Z}),
\]

where \( f'_1 w_1 + f'_2 w_2 = \delta' \) is any fixed solution. Since \( \inf_i \{|w_i|\} \leq B \), it follows that

1. If \( |w_1| \leq B \), then \(-B \leq u_1 + k f'_2 \leq B \) implies \(- (u_1 + B)/f'_2 \leq k \leq -(u_1 - B)/f'_2 \);
2. If \( |w_2| \leq B \), then \(-B \leq u_2 - k f'_1 \leq B \) implies \((u_2 - B)/f'_1 \leq k \leq (u_2 + B)/f'_1 \).

Thus, if \( f_1 f_2 \neq 0 \), then \( T(\delta, F, B) \) is bounded uniformly in \( \gamma \) since \( f'_1, f'_2 \geq 1 \) implies

\[
T(\delta, F, B) \leq \max \left\{ \frac{2B}{\delta'}, \frac{2B}{\delta'} + 2 \right\} \ll_{\mu} B.
\]

If, however, \( f_1 f_2 = 0 \), then we may assume, by symmetry, that \( f_2 = 0 \). Using the standard notation \( \tau(k) \) to denote the number of distinct divisors of an integer \( k \), it is then clear that

\[
T(\delta, F, B) = \# \{ (w_1, w_2) \in \mathbb{Z}^2 : f_1 w_1 = \delta \text{ and } \max_i \{|w_i|\} \leq B \}
= \# \{ (w_1, w_2) \in \mathbb{Z}^2 : w_1 \text{ divides } |\delta'| \text{ and } \max_i \{|w_i|\} \leq B \}
\leq B \tau(|\delta'|) \ll_{\varepsilon} B|\delta'| \ll_{\mu, \gamma, \varepsilon} B^{1+\varepsilon}.
\]
Combining these two cases into one inequality shows (5.10)

\[ T(\delta, F, B) \ll_{\mu, \nu, \gamma, \varepsilon} B^{1+\varepsilon}. \]

**Part 2:** We can now proceed by induction on \( n \). It is clear that the case \( n = 1 \) is true because \( \# \{ w_1 \in \mathbb{Z} : f_1 w_1 = \delta \text{ and } |w_1| \leq B \} \leq 2B \). The case \( n = 2 \) is dealt with by Part 1.

Now assume \( n \geq 3 \) and that Lemma 7 is true for \( n-1 \). We know that there exist \( i \) such that \( f_i \neq 0 \). Permuting indices, if needed, we may assume that \( f_n \neq 0 \). Let \( (w_1, \ldots, w_n) \in \mathbb{Z}^n \) satisfy \( \max(|w_1|, \ldots, |w_n|) \leq B \) and \( f_1 w_1 + \cdots + f_n w_n = \delta \).

Setting \( y := f_1 w_1 + \cdots + f_{n-1} w_{n-1} \), it follows that \( (y, w_n) \in \mathbb{Z}^2 \) satisfies the equation \( y + f_n w_n = \delta \) and, as well, the condition \( \inf(y, w_n) \leq B \). Thus, Part 1 implies that the number of such \( (y, w_n) \) is \( \ll_{\mu, \nu, \gamma, \varepsilon} B^{1+\varepsilon} \), uniformly over \( f_n \).

We can now also think of \( y \) as playing the role of \( \delta \). The evident bound

\[ |y| \leq |f_1| |w_1| + \cdots + |f_{n-1}| |w_{n-1}| \leq (n-1) \gamma B^{\mu+1} \]

then tells us that by replacing \( \gamma \) respectively \( \mu \) by \((n-1) \gamma \) respectively \( \mu + 1 \), we can apply the induction hypothesis, which yields the bound

\[ \# \{ (w_1, \ldots, w_{n-1}) \in \mathbb{Z}^{n-1} : \sum_{i=1}^{n-1} f_i w_i = y \text{ and } \max(|w_1|, \ldots, |w_{n-1}|) \leq B \} \ll_{\mu, \nu, \gamma, \varepsilon} B^{n-2+\varepsilon}, \]

uniformly in \( y, f_1, \ldots, f_{n-1} \) (clearly, each \( f_i \) continues to satisfy (iii) with \((n-1) \gamma \) as factor for \( B^{\mu+1} \)). Multiplying this bound with that in the preceding paragraph implies

\[ \# \{ (w_1, \ldots, w_n) \in \mathbb{Z}^n : f_1 w_1 + \cdots + f_n w_n = \delta \text{ and } \max(|w_1|, \ldots, |w_n|) \leq B \} \ll_{\mu, \nu, \gamma, \varepsilon} B^{1+\varepsilon} \cdot B^{n-2+\varepsilon} \ll_{\mu, \nu, \gamma, \varepsilon} B^{n-1+\varepsilon}. \]

This completes the proof of the lemma. 

**Note:** We use Lemma 7 in Step 3 to bound each \( N(\delta, z, \Delta) \) uniformly over \( \mathcal{Y}(\delta, z) \). Then, we use a uniform bound for the number of \( \Delta \) and (5.8) to give a bound for \( N_{\delta}(z) \) that is uniform in \( z \).

**Step 3:** To apply Lemma 7, we need to find the parameters \( \gamma, \mu \) so that conditions (ii), (iii) of the Lemma are satisfied. By Hadamard’s inequality, we know that any \( \delta \in \mathcal{D}(x) \) satisfies \( |\delta| \ll x^{n+1} \). Thus, we should set \( B = x \), which implies \( |\delta| \ll B^{n+1} \).

Similarly, each vector \( \Delta \in \mathcal{Y}(\delta, z) \) is a vector of determinants, each of whose components \( \Delta_{1,i} \), in absolute value, satisfies the bound

\[ |\Delta_{1,i}| \ll B^{n-1} \text{ uniformly in } z, i. \]

As a result, it follows that any \( \mu \), chosen so that condition (ii) in Lemma 7 is true, satisfies \( n < \mu < n + 1 \) uniformly in \( m, z, u \). In addition, any parameter \( \gamma \), chosen so that condition (iii) is satisfied, can also be chosen to be bounded by 1 uniformly in \( m, z, u \).

It is also clear that choosing \( B \) in this way implies that we can fix \( \nu = 1 \).

We can therefore eliminate any reference to the three parameters \( \mu, \nu, \gamma \), when we apply Lemma 7 because they are all bounded by constants that are uniform in \( z \). This helps simplify the rest of the discussion.
We can now write down the three estimates we need. The first one bounds the number of \( z \) in terms of \( x \)

\[
\# \mathcal{F}(x) \ll_{\theta_1} x^{\varepsilon_1 + \theta_1}.
\]

(5.11)

The number of possible coefficient vectors \( \Delta(m) \in Y(\delta, z) \) is bounded by the number of possible \( m' \in \prod_{2 \leq j \leq n} \mathcal{F}(x) \). Thus,

\[
\# \mathcal{Y}(\delta, z) \ll_{\theta_2} x^{n(\varepsilon_2 + \theta_2)} \quad \text{uniformly in } z.
\]

(5.12)

The discussion from Step 2 tells us that Lemma 7 gives the following bound for each \( N(\delta, z, \Delta) (\Delta \in \mathcal{Y}(\delta, z)) \):

\[
N(\delta, z, \Delta) \ll_{\theta_3} x^{n-1+\theta_3}.
\]

(5.13)

Combining (5.3), (5.8) with (5.11)–(5.13) now tells us that for any \( \delta \in D(x) \)

\[
N_\delta \ll_{\theta_1, \theta_2, \theta_3} x^{[n-1+ne_\delta + \theta_1 + \theta_2 + \theta_3]}.
\]

It follows that

\[
\|N(x)\| \ll_{\theta_1, \theta_2} x^{[n-1+ne_\delta + \theta_1 + \theta_2 + \theta_3]} \sqrt{D(x)}.
\]

(5.14)

As a result, by combining (5.6) and (5.14) we conclude

\[
A(f_4, x) \ll_{\theta_1, \theta_2, \theta_3} x^{[2n+n+1+ne_\delta + \theta_1 + \theta_2 + \theta_3]} \cdot D(x).
\]

(5.15)

Since each \( \theta \) has weight \( nD_\beta + \varepsilon_\theta \), the fact that

\[
A(f_4, x) \gg_{\theta} x^{nD_\beta + \varepsilon_\delta - \theta} = x^{\omega(Z) - \theta} \quad (Z = \zeta\text{sim}(\mathcal{F}, \hat{s})�)
\]

then implies, by a simple calculation left to the reader, if \( e_\delta > n - 1 \), then for any \( \varepsilon_1 > 0 \), we can find \( \theta, \theta_1, \theta_2, \theta_3 > 0 \) so that

\[
[nD_\beta + \varepsilon_\delta - \theta] - [2n + n + 1 + ne_\delta + \theta_1 + \theta_2 + \theta_3] > e_\delta - (n - 1) - \varepsilon_1 > 0.
\]

As a result, if \( e_\delta > n - 1, \varepsilon_1 \) is any positive number such that \( e_\delta > n - 1 + \varepsilon_1 \), and \( \varepsilon_2 \) is any positive number, then

\[
D(x) \gg_{\varepsilon_1} x^{e_\delta - n + 1 - \varepsilon_1} \gg_{\varepsilon_1, \varepsilon_2} [\# \mathcal{F}(x)]^{(e_\delta - n + 1 - \varepsilon_1)/(e_\delta + \varepsilon_2)} \quad \text{as } x \to \infty.
\]

As a result, given any \( \varepsilon > 0 \), we now choose \( \varepsilon_1, \varepsilon_2 \) so that the exponent of \( \# \mathcal{F}(x) \) on the right is \( > 1 - (n - 1)/e_\delta - \varepsilon \), which completes the proof of the asymptotic lower bound

\[
D(x) := \text{Vol}_{n, F}(x) \gg_{\varepsilon} [\# \mathcal{F}(x)]^{1-(n-1)/e_\delta - \varepsilon} \quad \text{as } x \to \infty.
\]

and finishes the proof of Theorem 4.

**Proof of Theorem 5.** This is an immediate application of Theorem 4 and the proof of Claim 2 (see §4.3), where we showed that \( P\text{as}(p) \) is thick and has upper Minkowski dimension \( e_p = \ln(p(p+1)/2)/\ln p > 1 \) (see [5]).

**Proof of Theorem 6.** This is an immediate application of Theorem 4 and the proof of Claim 3 (see §4.3), where we showed that \( \mathcal{M}_{n, p} \) is thick and has upper Minkowski dimension

\[
e_{n, p} = \frac{\ln \left( \left( \frac{p + n - 1}{p - 1} \right) \right)}{\ln p}.
\]

(5.16)

It is then clear that if (2.17) is satisfied, then \( e_{n, p} > n - 1 \). This allows us to apply Theorem 4. As a result, we conclude that for any sufficiently small \( \varepsilon > 0 \)

\[
\text{Vol}_{\mathcal{M}_{n, p}}(x) \gg_{\varepsilon} [\# \mathcal{M}_{n, p}(x)]^{1-(n-1)/e_{n, p} - \varepsilon} \quad \text{as } x \to +\infty.
\]

(5.16)
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