Zeta functions of discrete self similar sets and applications to Point Configuration and Sum-Product type problems*

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Abstract. In this paper we first prove analytical properties of zeta functions for discrete subsets of $\mathbb{R}^n$ that exhibit “self-similarity” with respect to an arbitrary finite set of (affine) similarities. We then show how such properties help solve Point Configuration resp. Sum-Product type problems over $\mathbb{Z}$. We do so by first extending a classic one variable Tauberian theorem of Ingham to several variables to derive a non trivial lower bound on the average of coefficients of an appropriate multivariate zeta function. We then combine this with well known results from Diophantine Geometry that prove uniform bounds for the density of lattice points in families of algebraic hypersurfaces.

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1 Introduction

In this article we apply zeta function methods for self similar subsets to detect solutions for a variety of Point Configuration and Sum-Product type problems over $\mathbb{Z}$. Unlike our earlier work [EL1]-[EL3], we only need a simple geometric overlap constraint on the sets to prove our results (see Definition 4).

In addition to this geometric condition, our prior work had also assumed the algebraic condition of “compatibility.” This required the underlying similarity transformations to pairwise commute. Here we eliminate the compatibility hypothesis. As a result, we are now able to solve such problems for a much larger class of self similar subsets of $\mathbb{Z}^n$.

We first show (see §2.1, §3) how the zeta functions for our class of self similar sets can be continued outside a halfplane of absolute convergence as a meromorphic function without assuming the compatibility condition on the similarities. This is a basic prerequisite for all our results, which are, to be clear, multivariate in nature.

We then prove (see §2.2, §4) a multivariate Tauberian theorem that is an extension of a classical one variable result of Ingham [I]. This is the main analytical tool of the article.

Our goal is to use this theorem to detect solutions of some typical “distinct value” distribution problems of interest in the combinatorial geometric study of self similar sets “in the large” [St] (i.e., “at infinity”). The values are those of different metric invariants determined by points of a self

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similar set in an increasing family of discs. The invariants of interest here are distance and angle for pairs of points, volumes of \( n \)-dimensional parallelopipes, and \( k \)-point configurations when \( k \geq 3 \).

The idea is that the Tauberian theorem helps find an explicit lower bound for a suitable average of the metric invariant of interest. This becomes possible by introducing a multivariate “fractal zeta function” whose coefficients encode the values of interest.

Our solutions to different distribution problems are stated in \( \S 2.3 \) and proved in \( \S 5 \). The underlying idea is both classic and quite simple to describe. We combine our lower bound from analysis with well known uniform upper bounds for lattice point counts on increasing families of nonsingular (quadric) hypersurfaces. The end result is a proof that the number of distinct metric invariant values must grow without bound if the upper Minkowski dimension of the self similar set is above a simple threshold.

In \( \S 2.4 \) and \( \S 6 \) we also apply this method to address the Erdös-Szemerdi conjecture for certain families of finite subsets of \( \mathbb{Z} \). This extends our preceding article \( [EL4] \) that showed how zeta functions can help solve the conjecture for certain families of finite subsets of a self similar set in \( \mathbb{Z} \). In this article, the finite sets need not be subsets of a self similar set. Instead, it suffices that each set belongs to the projection of a self similar subset of \( \mathbb{Z}^n \) \((n \geq 2)\) onto some coordinate axis.

**Notations:** Some notations of use in this paper are as follows:

1. \( \mathbb{N} = \{1, 2, \ldots \} \), \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \).
2. For a vector \( b = (b_1, \ldots, b_n) \in \mathbb{N}_0^n \), set \( |b| = b_1 + \cdots + b_n \); and \( y^b = \prod_j y_j^{b_j} \) for any \( y = (y_1, \ldots, y_n) \in \mathbb{R}^n \). For \( X = (X_1, \ldots, X_k) \in (\mathbb{R}^n)^k \) and \( \alpha = (\alpha_1, \ldots, \alpha_k) \in (\mathbb{N}_0^n)^k \), we will use also the notation \( X^\alpha = \prod_{j=1}^k X_j^{\alpha_j} \).
3. Given an unbounded variable \( x \in X \) and parameter \( y \in Y \), the expression
   \[
   f(y, x) \ll_y g(x) \quad \text{uniformly in } x \in X
   \]
   means that for any \( y \in Y \) there exists a constant \( C = C(y) > 0 \), such that for each \( x \in X \)
   \[
   |f(y, x)| \leq C \cdot g(x). \]
   If \( X = [1, \infty)^n \) we will express this property in an equivalent manner by writing
   \[
   f(y, x) \ll_y g(x) \quad \text{as } \inf_{1 \leq i \leq r} x_i \to \infty.
   \]
4. Let \( F(s) \) be a meromorphic function on a domain \( D \) of \( \mathbb{C} \) and let \( \mathcal{P} \) be the set of its poles. We define \((Res, Im s) = (\sigma, t)\). In addition, we say that \( F \) has moderate growth on \( D \) if there exists \( a, b > 0 \) such that \( \forall \theta > 0, F(s) \ll_{\sigma, \theta} 1 + |t|^{|\sigma|+b} \) uniformly in \( s = \sigma + it \in D \) such that \( d(s, \mathcal{P}) \geq \theta \);
5. Fixing an orthonormal basis \( \mathcal{B} = \{e_1, \ldots, e_n\} \) of \( \mathbb{R}^n \), we do not distinguish between a polynomial \( H(X) \in \mathbb{C}[X_1, \ldots, X_n] \) and the polynomial function on \( \mathbb{R}^n \) determined by the basis \( \mathcal{B} \)
   \[
   x_1e_1 + \cdots + x_ne_n \mapsto H(x_1, \ldots, x_n);
   \]
6. Let \( f \) be a meromorphic function in a domain \( U \) of \( \mathbb{C} \) and \( a \in U \). If \( a \) is a pole of \( f \) we denote by \( ord_{s=a} f(s) \) (resp. \( Res_{s=a} f(s) \)) the order (resp. the residue) of \( a \) as a pole of \( f \).

By convention we set \( ord_{s=a} f(s) = 0 \) and \( Res_{s=a} f(s) = 0 \) if \( f \) is regular in \( a \).

## 2 Preliminaries and statements of main results

### 2.1 Self-similar discrete sets and their zeta functions

*Throughout this article, we restrict attention to unbounded discrete subsets of \( \mathbb{R}^n \) \((n \geq 1)\), and will*
always denote such a set by $F$.

**Definition 1.** A similarity on $\mathbb{R}^n$ is an affine transform $f : \mathbb{R}^n \to \mathbb{R}^n$ of the form $f(x) = cT(x) + b$, where $c = c(f) > 0$ is the scale factor of $f$ and $T = T(f)$ is an orthogonal transformation of $\mathbb{R}^n$ with respect to the quadratic form $\|x\|^2 = \sum_i x_i^2$.

**Definition 2.** $F$ is a self similar set if there exists a finite set $f = \{f_i = c_iT_i + b_i\}_{i=1}^d$ of affine similarities such that each scale factor $c_i > 1$ and $F \equiv \bigcup_{i=1}^d f_i(F)$.

We next recall the notion of dimension that suffices for our purposes.

**Definition 3.** The upper Minkowski dimension of $F$ equals:

$$\text{udim}_M F := \lim_{R \to \infty} \frac{\ln \left| \#F(R) \right|}{\ln R} \in [0, \infty)$$

where $F(R) := F \cap B(R) = \{m \in F; \|m\| \leq R\}$.

**Note.** We will subsequently restrict attention to discrete $F$ whose upper Minkowski dimension is finite. For the applications in §2.3, we also need to assume this dimension is positive. We denote the upper Minkowski dimension of $F$ by $e_F$.\(\square\)

By specifying the sizes of overlaps of the sets invariant under the individual $f_i$, two distinct possibilities occur for any self similar set.

**Definition 4.** We say that

1. $F$ satisfies the “finite overlap” condition if

$$f_i(F) \cap f_j(F) \text{ is finite if } i \neq j.$$  \hfill (1)

2. $F$ satisfies the “normal overlap” condition if

$$e'_F := \max \{\text{udim}_M (f_i(F) \cap f_j(F))\} < e_F \text{ for all } i \neq j.$$  \hfill (2)

**Remark.** The finite overlap condition (1) can be understood to be a reasonable analogue of the open set condition for compact fractals due to Moran [M] since outside a bounded subset the different invariant subsets must be disjoint.

Of course, many other possibilities can occur for the density of overlaps of different invariant subsets. We have chosen these two since they reflect opposite extremes of what can happen in general. In addition, it is worth emphasizing that the zeta function method appears to be useful not only for these two cases but for others as well, though we do not address this point here. \(\square\)

The zeta function of $F$ is the Dirichlet series defined by

$$\zeta_F(s) := \sum_{m \in F'} \|m\|^{-s} \quad \text{where} \quad F' := F - \{0\}.$$  \hfill (3)

For several applications we will also need to study a “weighted” zeta function, defined by choosing $\alpha \in \mathbb{N}_0^n$ and setting

$$\zeta_F(s, \alpha) := \sum_{m \in F'} \frac{m^\alpha}{\|m\|^s}.$$  \hfill (4)

It follows from an elementary estimation (see [EL1]) that for any $\alpha \in \mathbb{N}_0^n$

$$\zeta_F(s, \alpha) \text{ converges absolutely in the half-plane } \{\sigma > e_F + |\alpha|\}.$$  \hfill (5)

\(3\)The notation $F \equiv G$ means that $(F \setminus G) \cup (G \setminus F)$ is a finite set.
Definition 5. Let $f = \{f_j = c_j T_j + b_j\}_{j=1}^d$ be a finite set of affine similarities.

Let $r \in \mathbb{N}_0$. Set $N_r = \binom{n+r}{n-1}$ and denote by $\alpha_1, \ldots, \alpha_N$ the elements $\alpha_i = (\alpha_{1,i}, \ldots, \alpha_{n,i}) \in \mathbb{N}_0^n$ of weight $r$ and ordered lexicographically.

The transformation $y = (y_1, \ldots, y_n) \to x = T_j(y)$ induces a linear map on the vector space of monomials of degree $r$

\[ x^\alpha = \sum_{v=1}^{N_r} g^{(j)}_{(u,v)} y^\alpha. \]  

(6)

We define the $N_r \times N_r$ matrix $M_r(s)$ $(s \in \mathbb{C})$, where $M_r(s)_{(u,v)} = \sum_{j=1}^{d} g^{(j)}_{(u,v)} \epsilon_j^{-s}$;

and set

\[ \delta_r(s) := \det(I_{N_r} - M_r(s)). \]  

(7)

Remark 1. If $r = 0$, then $\delta_0(s) = 1 - \sum_{j=1}^{d} \epsilon_j^{-s}$ is the “Dirichlet polynomial” for $F$. Its roots are fairly easy to understand since the coefficient of each $\epsilon_j^{-s}$ has the same sign. When $r \geq 1$, however, this property will not in general occur for $\delta_r(s)$. As a result, it is more difficult to study its roots.

We first describe the behavior of $\zeta_F(s)$ outside the half-plane $\{\sigma > e_F\}$ of absolute convergence.

Theorem 1. Let $F \subset \mathbb{R}^n$ be a self-similar set, determined by a set of similarities $f$ as in Definition 2. Then the following properties are satisfied.

1. $e_F$ is the abscissa of convergence of $\zeta_F(s)$, and equals the unique real solution of the equation $\delta_0(s) = 0$.

2. If the finite overlap condition (1) holds, then $\zeta_F(s)$ has a meromorphic extension with moderate growth to the whole complex plane $\mathbb{C}$.

3. If the normal overlap condition (2) holds, then $\zeta_F(s)$ has a meromorphic extension with moderate growth to the half plane $\{\sigma > e'_F\}$.

4. In each case:

   (a) the polar locus $\mathcal{P}ol_F$ of $\zeta_F(s)$ is a subset of

   \[ \{\sigma \leq e_F\} \cap \bigcup_{r \in \mathbb{N}_0} \bigcup_{k \leq r} \{s + k : \delta_r(s) = 0\}; \]  

   (8)

   (b) $s \mapsto \delta_0(s) \cdot \zeta_F(s)$ is holomorphic and has moderate growth in the half-plane $\{\sigma > e''_F\}$, where $e''_F = e_F - 1$ if (1) holds, and $e''_F = \max(e_F - 1, e'_F)$ if (2) holds.

   (c) $e_F$ is a simple pole of $\zeta_F(s)$.

Theorem 1 is a particular consequence of the following more general result that is proved in §3.1.

Theorem 2. Let $F$ be a self similar set. Let $\alpha$ have weight $|\alpha| = r \geq 0$. The following three properties are satisfied.

1. If the finite overlap condition (1) holds, then $\zeta_F(s, \alpha)$ has a meromorphic extension with moderate growth to the whole complex plane $\mathbb{C}$.
2. If the normal overlap condition (2) holds, then \( \zeta_F(s, \alpha) \) has a meromorphic extension with moderate growth to the half-plane \( \{ \sigma > e_F + r \} \).

3. In each case:

   (a) the polar locus \( \mathcal{P} \text{ol}_F(\alpha) \) of \( \zeta_F(s, \alpha) \) is a subset of

   \[
   \{ \sigma \leq e_F + r \} \cap \bigcup_{q \in \mathbb{N}_0} \bigcup_{k \leq q} \{ s + k : \delta_q(s) = 0 \};
   \]

   (b) \( s \mapsto \delta_r(s - r) \cdot \zeta_F(s, \alpha) \) is holomorphic and has moderate growth in the half-plane

   \( \{ \sigma > e'_F + r \} \), where \( e'_F = e_F - 1 \) if (1) holds, and \( e'_F = \max(e_F - 1, e'_F) \) if (2) holds.

**Notation Convention.** We subsequently use the notations \( \zeta_F(s), \zeta_F(s, \alpha) \) to denote both the Dirichlet series and their meromorphic extensions.

### 2.2 A multivariate extension of Ingham’s Tauberian theorem

We begin with a self-similar subset \( F \) of \( \mathbb{R}^n \) satisfying one of the overlap conditions from Definition 4. We also consider a multivariate polynomial \( R \) with real coefficients of the form

\[
R(X_1, \ldots, X_k) = \sum_{\alpha \in \text{Supp}(R)} b(\alpha) \prod_{j=1}^{k} X_j^{\alpha_j}.
\]

**Notation.** For simplicity, we will always use the notation \( \alpha \) to denote a \( k \)-vector \( (\alpha_1, \ldots, \alpha_k) \in (\mathbb{N}_0)^k \), and we define \( \text{Supp}(R) = \{ \alpha : b(\alpha) \neq 0 \} \). Throughout the discussion below, the notation \( \alpha_j \) always denotes the \( j \)th component of a vector \( \alpha \in \text{Supp} R \).

We first define the density for any \( x = (x_1, \ldots, x_k) \in (0, \infty)^k \) by setting:

\[
A(R, x) = A(R, x_1, \ldots, x_k) := \sum_{0 < |m_1| \leq s_1, \ldots, \alpha_j, \ldots, \alpha_k} R(m_1, \ldots, m_k).
\]

In addition, we also define the multivariable zeta function formally for \( s = (s_1, \ldots, s_k) \in \mathbb{C}^k \) by setting:

\[
Z(R, s) = Z(R, s_1, \ldots, s_k) := \sum_{m_1, \ldots, m_k \in F} \frac{R(m_1, \ldots, m_k)}{|m_1|^{s_1} \cdots |m_k|^{s_k}}.
\]

It follows from \( \S 2.1 \) that \( Z(R, s) \) converges absolutely and defines a holomorphic function in the domain

\[
\mathcal{D} = \bigcap_{\alpha \in \text{Supp} R} \{ s \in \mathbb{C}^k : \sigma_j > e_F + |\alpha_j| \ \forall j = 1, \ldots, k \},
\]

where it evidently satisfies the identity

\[
Z(R, s) = \sum_{\alpha \in \text{Supp}(R)} b(\alpha) \cdot \prod_{j=1}^{k} \zeta_F(s_j, \alpha_j).
\]

For each \( \alpha \in \text{Supp}(R) \) we set \( \tilde{\alpha} = (|\alpha_1|, \ldots, |\alpha_k|) \) and define

1. \( \text{Supp}_0(R) := \{ \alpha \in \text{Supp}(R) : |\tilde{\alpha}| := \sum_{j=1, \ldots, k} |\alpha_j| = \text{deg}(R) \} \);
2. \( E(R) := \{ \alpha : \alpha \in \text{Supp}_0(R) \} \subset \mathbb{N}_0^k \),
as well as

\[
\text{the polyhedron } \Gamma(R) = \text{the boundary of the convex hull of } E(R); \\
\text{the set of vertices } \mathcal{V}(R) \text{ of } \Gamma(R).
\]

For any \( \mathbf{v} \in E(R) \) we also define:

\[
\text{Supp}(R, \mathbf{v}) := \{ \alpha \in \text{Supp}_0(R) : \mathbf{\alpha} = \mathbf{v} \};
\]

\[
R_\mathbf{v}(\mathbf{x}_1, \ldots, \mathbf{x}_k) = \sum_{\alpha \in \text{Supp}(R, \mathbf{v})} b(\alpha) \mathbf{x}_1^{\alpha_1} \cdots \mathbf{x}_k^{\alpha_k};
\]

\[
Z(R_\mathbf{v}, \mathbf{s}) = \sum_{\mathbf{m}_1, \ldots, \mathbf{m}_k \in \mathbb{F}} \frac{R_\mathbf{v}(\mathbf{m}_1, \ldots, \mathbf{m}_k)}{||\mathbf{m}_1||^{s_1} \cdots ||\mathbf{m}_k||^{s_k}} = \sum_{\alpha \in \text{Supp}(R, \mathbf{v})} b(\alpha) \prod_{j=1}^{k} \zeta_\mathbf{F}(s_j, \alpha_j).
\]

We know from (5) and Theorem 2 that for each \( \mathbf{v} = (v_1, \ldots, v_k) \in E(R) \), \( \alpha = (\alpha_1, \ldots, \alpha_k) \in \text{Supp}(R, \mathbf{v}) \), and \( j = 1, \ldots, k \):

\( \zeta_\mathbf{F}(s_j, \alpha_j) \) converges absolutely for \( \{ \sigma_j > e_F + v_j \} \), and has a meromorphic extension to the left of this half-plane.

This determines the meromorphic extension of the Dirichlet series \( Z(R_\mathbf{v}, \mathbf{s}) \) whose iterated residue values will be crucial for the remaining discussion.

Denoting the residue of \( \zeta_\mathbf{F}(s_j, \alpha_j) \) at \( s_j = e_F + v_j \) by \( \text{Res}_{s_j = e_F + v_j} \zeta_\mathbf{F}(s_j, \alpha_j) \), we define, for any \( \mathbf{v} \in E(R) \), the iterated residue of \( Z(R_\mathbf{v}, \mathbf{s}) \) at \( \mathbf{s} = e_F \mathbf{1}_k + \mathbf{v} \) by

\[
\text{Res}_{\mathbf{s} = e_F \mathbf{1}_k + \mathbf{v}} Z(R_\mathbf{v}, \mathbf{s}) := \sum_{\alpha \in \text{Supp}(R, \mathbf{v})} b(\alpha) \prod_{j=1}^{k} \text{Res}_{s_j = e_F + v_j} \zeta_\mathbf{F}(s_j, \alpha_j).
\]

Our principal result is the following multivariate Tauberian theorem.

**Theorem 3.** Assume \( R|_{x^k} \geq 0 \) and that for any vertex \( \mathbf{v} \in \mathcal{V}(R) \):

\[
R_\mathbf{v}|_{x^k} \geq 0 \quad \text{and} \quad \text{Res}_{\mathbf{s} = e_F \mathbf{1}_k + \mathbf{v}} Z(R_\mathbf{v}, \mathbf{s}) \neq 0.
\]

Then, for any sufficiently small \( \delta > 0 \),

\[
A(R, x, \ldots, x) := \sum_{\substack{\mathbf{m}_1, \ldots, \mathbf{m}_k \in \mathbb{F} \\text{with} \ \mathbf{m}_j \leq x, \ \forall j = 1, \ldots, k \ \text{and} \ ||\mathbf{m}_j|| \leq x^{\delta}}} R_\mathbf{v}(\mathbf{m}_1, \ldots, \mathbf{m}_k) \geq x^{k_{e_F} - \text{deg } R - \delta} \quad \text{as } x \to \infty.
\]

The main ingredient in the proof is a description of the asymptotic behavior, for any \( \mathbf{v} \in E(R) \), of the weighted average of \( R_\mathbf{v} \) values:

\[
A_\mu(R_\mathbf{v}, x_1, \ldots, x_k) := \sum_{\mathbf{m}_1, \ldots, \mathbf{m}_k \in \mathbb{F} \ \text{with} \ \mathbf{m}_j \leq x_j, \ \forall j = 1, \ldots, k \} R_\mathbf{v}(\mathbf{m}_1, \ldots, \mathbf{m}_k) \prod_{j=1}^{k} ||\mathbf{m}_j||^{\mu_j} \ \forall \mu = (\mu_1, \ldots, \mu_k) \in \mathbb{R}_+^k.
\]

**Theorem 4.** Let \( \mathbf{v} \in \mathcal{V}(R) \). Assume that \( R_\mathbf{v}|_{x^k} \geq 0 \) and \( A_0(R_\mathbf{v}) := \text{Res}_{\mathbf{s} = e_F \mathbf{1}_k + \mathbf{v}} Z(R_\mathbf{v}, \mathbf{s}) \neq 0 \). Then

1) \( A_{e_F \mathbf{1}_k + \mathbf{v}}(R_\mathbf{v}, x_1, \ldots, x_k) = A_0(R_\mathbf{v}) \cdot \left( \log x_1 \cdots \log x_k \right) \cdot \left( 1 + O\left( \sum_{j=1}^{k} \frac{1}{\log(\log x_j)} \right) \right) \)

as \( \inf_{1 \leq j \leq k} x_j \to \infty. \)
2) For any \( \mu \in \prod_{j=1}^{k} [0, e^F + v_j] \) and any \( \kappa > 0 \):

\[
A_\mu(R_v, x_1, \ldots, x_k) \gg \prod_{j=1}^{k} x_j^{e^F + v_j - \mu_j - \kappa} \quad \text{as} \quad \inf_{1 \leq j \leq k} x_j \to \infty.
\]  

(20)

### 2.3 Solutions of point configuration problems over \( \mathbb{Z} \) for distinct distances, angles, \( k \)-configurations, \( k \)-chains, and volumes

We use Theorems 3 and 4 to prove five explicit lower bound results in §5 whose statements are given below. Each is asymptotic in nature in the sense that they hold for all sufficiently large values of a parameter \( x \). The metric invariants of interest to us specify distance, angles, \( k \)-configurations, \( k \)-chains, and volumes determined by finite sets of points that belong to any self similar subset \( \mathcal{F} \subset \mathbb{Z}^n \) satisfying one of the two overlap properties from Definition 4.

To simplify matters, we will call such sets “suitable”.

#### 2.3.1 Lower bounds for distinct distances

For any \( x > 0 \), we first define (see Definition 3 and (3))

\[
\text{Dist}_\mathcal{F}(x) := \# \{ \| \mathbf{m}_1 - \mathbf{m}_2 \| : \mathbf{m}_1, \mathbf{m}_2 \in \mathcal{F}'(x) \},
\]

(21)

with the understanding that \( \text{Dist}_\mathcal{F}(x) \) counts each distance exactly once. In §5.1 we prove the following.

**Theorem 5.** Assume \( \mathcal{F} \) is a suitable set with upper Minkowski dimension \( e_F > n - 2 \). Then for any sufficiently small \( \varepsilon > 0 \),

\[
\text{Dist}_\mathcal{F}(x) \gg \varepsilon \left[ \# \mathcal{F}(x) \right]^{1 - \frac{n-2}{n^2 - 2} - \varepsilon} \quad \text{as} \quad x \to \infty .
\]

(22)

**Remark 2.** If \( n = 2 \), it seems worth remarking that our lower bound (22), which holds for any self similar subset of \( \mathbb{Z}^2 \), satisfying either (1) or (2), is only a bit worse than what follows from the celebrated (and much more general) work of Guth-Katz [GK]. Moreover, we should also emphasize that when \( n > 2 \) and \( e_F \) satisfies

\[
n \cdot \frac{n^2 - 4}{n^2 - 2} < e_F \leq n ,
\]

the exponent of \( \# \mathcal{F}(x) \) is larger than that obtained in [SV], which was also proved in much greater generality. The point is that if \( e_F \) is in the above interval, then the asymptotic result we can prove gives a larger exponent than that which follows from [ibid.].

#### 2.3.2 Lower bounds for distinct angles

In addition to the distance between any pair of points of \( \mathcal{F}' \), it is also natural to ask about the distribution of angles formed by any pair with a fixed origin. To this end, for \( \mathbf{m}_1, \mathbf{m}_2 \in \mathcal{F}' \) we set:

\[
\theta(\mathbf{m}_1, \mathbf{m}_2) = \text{the angle formed between } \frac{\mathbf{m}_1}{\| \mathbf{m}_1 \|} \text{ and } \frac{\mathbf{m}_2}{\| \mathbf{m}_2 \|},
\]

(23)

where we think of the two unit vectors as belonging to the unit sphere with center the origin \( \mathbf{0} \). Defining

\[
\text{Ang}_\mathcal{F}(x) = \# \{ \cos(\theta(\mathbf{m}_1, \mathbf{m}_2)) : \mathbf{m}_1, \mathbf{m}_2 \in \mathcal{F}'(x) \},
\]

(24)

we prove in §5.2 the following.
Theorem 6. With the same hypotheses as in Theorem 5, if \( n \geq 4 \), then for any \( \varepsilon > 0 \):
\[
\text{Ang}_F(x) \gg \varepsilon \left[ \# F(x) \right]^{1 - \frac{n+2}{n-2} - \varepsilon} \quad \text{as } x \to \infty.
\] (25)

2.3.3 Lower bounds for \( k \)-point configurations and \( k \)-chains

We prove in §5.3 a natural extension of Theorem 5 to both “rooted \( k \)-configurations” and “\( k \)-chains”. These reflect different types of configurations determined by distances between subsets of \( k+1 \) points of discrete sets. Problems about the sizes of such sets have been studied in the context of bounded fractal like sets of positive Hausdorff dimension, but comparable questions for discrete sets have received less attention. Our method allows both of these configuration problems to be solved in essentially the same way.

For \( k \geq 2 \), define
\[
\text{Root}_{k,F}(x) := \# \left\{ (\|m_1 - m_{k+1}\|, \ldots, \|m_k - m_{k+1}\|) : m_j \in F'(x) \forall j \text{ and } m_j \neq m_{k+1} \forall j \leq k \right\}.
\] (26)

Theorem 7. Assume the same hypotheses as in Theorem 5. Then for any sufficiently small \( \varepsilon > 0 \),
\[
\text{Root}_{k,F}(x) \gg \varepsilon \left[ \# F(x) \right]^{k \left[ 1 - \frac{n+2}{n-2} - \varepsilon \right]} \quad \text{as } x \to \infty.
\] (27)

An immediate consequence of the proof of Theorem 7 is an extension to arbitrary \( k \)-point configurations. Define
\[
\text{Config}_{k,F}(x) := \# \left\{ (\|m_1 - m_{k+1}\|, \ldots, \|m_k - m_{k+1}\|, \|m_1 - m_k\|, \ldots, \|m_{k-1} - m_k\|, \ldots, \|m_1 - m_2\|) : m_j \in F'(x) \forall j \text{ and } m_i \neq m_j \forall i < j \right\}.
\] (28)

Corollary 1. Assume the same hypotheses as in Theorem 5. Then for any sufficiently small \( \varepsilon > 0 \),
\[
\text{Config}_{k,F}(x) \gg \varepsilon \left[ \# F(x) \right]^{k \left( 1 - \frac{n+2}{n-2} - \varepsilon \right)} \quad \text{as } x \to \infty.
\] (29)

A different type of configuration is given by a \( k \)-chain determined by \( k+1 \) points in \( F \). Define
\[
\text{Chain}_{k,F}(x) = \# \left\{ (\|x_1 - x_2\|, \|x_2 - x_3\|, \ldots, \|x_k - x_{k+1}\|) : (x_1, \ldots, x_{k+1}) \in F'(x)^{k+1} \text{ and } x_i \neq x_{i+1} \forall i = 1, \ldots, k \right\}.
\] (30)

The study of \( \text{Chain}_{k,F}(x) \) in the context of bounded fractal sets has been studied in [BIT] and [GIP]. The next result is, to our knowledge, the first that focuses upon the discrete case.

Theorem 8. Assume the same hypotheses as in Theorem 5. Then for any sufficiently small \( \varepsilon > 0 \),
\[
\text{Chain}_{k,F}(x) \gg \varepsilon \left[ \# F(x) \right]^{k \left( 1 - \frac{n+2}{n-2} - \varepsilon \right)} \quad \text{as } x \to \infty.
\] (31)
2.3.4 Lower bounds for distinct volumes of $n$–paralleloptopes

For any vector $m = (m_1, \ldots, m_{n+1}) \in (F')^{n+1}$ and $n$–paralleloptope $\Sigma_n(m)$ whose endpoints are $m_1, \ldots, m_{n+1}$, we denote the volume

$$|\Sigma_n(m)| = |\text{det}(m_1 - m_{n+1}, \ldots, m_n - m_{n+1})|,$$

and set

$$Vol_{n,F}(x) := \#\{|\Sigma_n(m)| : m \in F'(x)^{n+1}\}$$

(31)
to denote the number of distinct volumes of $n$–paralleloptopes. We first define in Definition 6 (see §5.4) what we mean by “thickness” of a self similar set, and then prove the following result:

**Theorem 9.** Assume that $F \subset \mathbb{Z}^n$ is thick, and $\epsilon > n - 1$. Then for sufficiently small $\epsilon > 0$:

$$Vol_{n,F}(x) \gtrsim \#F(x)^{1 - \frac{\epsilon}{n+1} - \epsilon} \quad \text{as} \quad x \to +\infty. \quad (32)$$

**Remark.** In [EL3] we gave some examples of thick self similar subsets, and in the preprint [EL5] we showed how, in a certain sense, thick self similar sets are “asymptotically generic”. Taken together this gives some evidence that thickness is a not uncommon feature of self similar sets.

2.4 Application to Sum-product estimates for self similar subsets of $\mathbb{Z}^n$.

Let $A$ be a finite subset of an abelian group $G$ and $k$ a positive integer. Define the $k$–fold sum set $kA$ and the $k$–fold product set $A^k$ of $A$ by:

$$kA = \{m_1 + \cdots + m_k \mid m_1, \ldots, m_k \in A\} \quad \text{and} \quad A^k = \{m_1 \cdots m_k \mid m_1, \ldots, m_k \in A\}.$$

In 1983 Erdős-Szemer edi [ES] conjectured that for subsets of integers, the sum set and the product set cannot both be small. Precisely, the conjecture asserts that for any $\epsilon > 0$, there exists $n_0$ such that for any finite set $A \subset \mathbb{Z}$ with $|A| \geq n_0$,

$$\max(|kA|, |A^k|) \gtrsim |A|^{k-\epsilon}. \quad (33)$$

Since the conjecture was first published, much work has been devoted to extending its scope to other fields or rings, as well as to proving partial results with a smaller exponent.

In [EL4], we used a “zeta function + Tauberian” method to prove the conjecture for any family of finite sets $F(x)$ determined by a self similar set $F \subset \mathbb{Z}$ satisfying overlap condition (1).

The result, stated in §2.4.2 and proved in §6, extends [EL4] to subsets of $\mathbb{Z}$ that are not themselves self similar but are the projections of self similar subsets in $\mathbb{Z}^n$ ($n \geq 2$) onto some coordinate axis. Our result shows that if the upper density of $F$ is sufficiently close to $n$ then at least one projection $\pi_i : F \to \mathbb{Z}$ exists so that the conjecture is satisfied for the subsets $\pi_i F \cap [-x, x]$ for all sufficiently large $x$. As such, this result is an analogue of our improvement of the result of Solymosi-Vu [SV] for the Erdős distance conjecture in $\mathbb{R}^n$ when $n \geq 3$ (see Remark 2) in that we are able to improve upon earlier results simply by controlling the upper density of a self similar subset of $\mathbb{Z}^n$.

In addition, by using a deep result of T. Browning and D.R. Heath-Brown [BHB] in Diophantine Geometry, we state in §2.4.3 and also prove in §6 a fairly general lower bound on the cardinality of sets $PF(x)$ when $F$ is a self similar subset of $\mathbb{Z}^n$ ($n \geq 2$) and $P$ is a homogeneous polynomial on $\mathbb{R}^{kn}$ for some $k \geq 1$. This result can be understood as an extension of Theorem 2 in [EL4] that was proved when $F \subset \mathbb{Z}$. Unlike the situation in §2.3, $P$ need not define a metric invariant on $\mathbb{R}^{kn}$.

We begin the discussion with §2.4.1 which states a needed preliminary result.
2.4.1 A lower bound for $|F(x)^\alpha|$ when $F \subset \mathbb{Z}^n (n \geq 2)$.

Given a self similar set $F \subset \mathbb{Z}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_k) \in (\mathbb{N}_0)^k$, we first define the set of monomial values determined by the points in any $F(x)$

$$F(x)^\alpha := \left\{ \prod_{j=1}^k m_j^{\alpha_j} : m_j \in F(x) \forall j = 1, \ldots, k \right\}.$$ 

**Theorem 10.** Let $F$ be a self-similar subset of $\mathbb{Z}^n$ satisfying overlap conditions (1) or (2). Assume that

$$\text{Res}_{s=\epsilon x+2|\alpha_j|\xi_{F}(s, 2\alpha_j)} \neq 0 \quad \forall j = 1, \ldots, k.$$ 

Then, for any $\epsilon > 0$,

$$|F(x)^\alpha| \gg_\varepsilon |F(x)|^{k - \frac{u(\alpha)}{\epsilon} - \varepsilon}$$

where $u(\alpha) = \{|(j, i) \in \{1, \ldots, k\} \times \{1, \ldots, n\}; \alpha_{j, i} = 0\}$.

2.4.2 Sum-Product estimate for projections of a self-similar subset of $\mathbb{Z}^n$

For each $1 \leq i \leq n$, we denote by $\pi_i : \mathbb{R}^n \to \mathbb{R}$ the projection defined by $\pi_i(x_1, \ldots, x_n) = x_i$. For any finite subset $A$ of $\mathbb{Z}^n$ and each $i$, we denote by $k(\pi_i A)$ and $(\pi_i A)^k$ the $k$–fold sum and product sets of $\pi_i A$.

The Erdős-Szemerédi sum-product conjecture [ES], when applied to the different $\pi_i A$, states that for subsets of integers, the $k$–sum and $k$–product sets of $\pi_i A$ cannot both be small for any $i$ for which this set is sufficiently large. Precisely, the conjecture asserts: For any $\varepsilon > 0$, there exists $M_0$ such that for any finite set $A \subset \mathbb{Z}^n$

$$|A| \geq M_0 \implies \max\{|k(\pi_i A)|, |(\pi_i A)^k|\} \gg_\varepsilon |\pi_i A|^{1-\varepsilon}$$

for each $i \in \{1, \ldots, n\}$.

A large number of partial results have been proved, each with a certain improvement in an exponent that remains rather far from “$k - \varepsilon$”. As of 2016, the best exponent we know of when $k = 2$ is due to Konyagin-Shkredov [KS]. Its value equals $1 + \frac{1}{3} + \frac{5}{4813} := 1 + \kappa_{KS}$.

We prove in §6 the following:

**Theorem 11.** Let $F$ be a self-similar subset of $\mathbb{Z}^n$ satisfying (1) or (2) and let $k \geq 2$. Then:

$$\exists i \in \{1, \ldots, n\} \text{ such that } \forall \varepsilon > 0 \quad \max\left\{|k(\pi_i F(x))|, |(\pi_i F(x))^k|\right\} \gg_\varepsilon |\pi_i F(x)|^{k(e_F+1-n) - \varepsilon}.$$ 

**Remarks:**

1. The estimate (36) is not trivial (i.e. the exponent is $> 1$) if $e_F > n - \frac{1}{2} + \varepsilon$.

2. $k = 2$ and $e_F > n - \frac{1}{2} + \frac{\kappa_{KS}}{2}$ imply our bound is better than that of Konyagin-Shkredov for at least one family of sets $\pi_i F(x)$ when $x \gg 1$.

3. Given any $k$ and any $\varepsilon > \varepsilon_1 > 0$, if $e_F > n - \frac{\varepsilon_1}{k}$, then $k(e_F + 1 - n) - \varepsilon_1 > k - \varepsilon$. In other words, starting with $\varepsilon$ if $e_F$ satisfies $e_F > n - \frac{\varepsilon_1}{k}$, then the exponent of the lower bound in (36) is at least the conjectured exponent. In other words, if $e_F$ is large enough, then the Erdős-Szemerédi conjecture holds for at least one infinite family of sets $\pi_i F(x)$ that need not belong to a self similar set. 

\[\square\]
2.4.3 Lower bound for distinct values of a polynomial restricted to a self similar subset of $\mathbb{Z}^n$

We give here a reasonably natural extension of Theorem 2 in [EL4]. We set $X_i = (X_{i1}, \ldots, X_{in})$ for $i = 1, \ldots, k$, $\alpha = (\alpha_1, \ldots, \alpha_k) \in (\mathbb{N}_0^n)^k$, and fix a polynomial $P \in \mathbb{Z}[X_1, \ldots, X_k]$ of degree $d > 0$ with integer coefficients of the form

$$P(X_1, \ldots, X_k) = \sum_{\alpha \in \text{Supp}(P)} a(\alpha) X_1^{\alpha_1} \cdots X_k^{\alpha_k} \quad (X_i^{\alpha_i} = \prod_{j=1}^n X_{ij}^{\alpha_{ij}} \forall i),$$

where $\alpha \in \text{Supp}(P) \subset (\mathbb{N}_0^n)^k$ iff $a(\alpha) \neq 0$.

Define for any $x > 0$ the $P$-fold sumset $PF(x)$ by

$$PF(x) = \{P(m_1, \ldots, m_k); \ m_j \in F'(x) \forall j = 1, \ldots, k\}.$$ 

The examples of interest in §2.3 are characterized by their invariance under orthogonal maps. There may be, however, some value in seeing what can be said when $P$ is a more general homogeneous polynomial.

The point of the following result is that a non trivial lower bound holds for $\#PF(x)$, provided that $P$ is a nonsingular form and $e_F$ is sufficiently large. This result uses in an essential way a strong uniformity property for the densities of the different level sets $\{P = t\} \cap \mathbb{Z}^n$ that is due to Browning-Heath-Brown [BHB].

**Theorem 12.** Let $k \geq 1$ and assume $F$ is a self-similar subset of $\mathbb{Z}^n$ satisfying both (1) or (2) and the property $e_F > n - \frac{1}{k}$.

Assume that $P \in \mathbb{Z}[X_1, \ldots, X_k]$ is a form of degree $d \geq 1$ that defines a non singular projective hypersurface $\{P = 0\} \subset \mathbb{P}^{kn-1}(\mathbb{Q})$.

Assume, in addition, that for each $j = 1, \ldots, k$,

$$\sum_{\alpha, \beta \in S_j(P)} a(\alpha)a(\beta) \text{Res}_{s=e_F+2d}\zeta_F(s; \alpha_j + \beta_j) \neq 0,$$

where $S_j(P) := \{\alpha \in \text{Supp}(P) : |\alpha_j| = d\}$.

Then:

$$\forall \varepsilon > 0 \quad |PF(x)| \gg \varepsilon |F(x)| \frac{\log \frac{k(e_F-n)+1}{e_F}}{e_F - \varepsilon}.$$ 

3 Proofs of Theorems 1 and 2

3.1 Proof of Theorem 2

To begin, we assume that either condition (1) or (2) holds.

We first introduce the basic objects needed for our proof by induction. These are a family of spaces of functions with a natural notion of degree, and a family of multiplication operators. The degree serves as a quantity with which a proof by induction can be carried out.

We set:
Lemma 2. Let \( h(\mathbf{X}) = \sum_{m \in \mathcal{M}} R(m) \| \mathbf{X} \|^s \) and \( k \in \mathbb{N}_0 \), and for any \( R = \frac{h(\mathbf{X})}{\| \mathbf{X} \|^k} \in \mathcal{H} \), define

\[
\deg R = \deg h - k \quad \text{and} \quad \zeta_{\mathcal{F}}(s, R) = \sum_{m \in \mathcal{F}} \frac{R(m)}{\| m \|^s} = \sum_{m \in \mathcal{F}} \frac{h(m)}{\| m \|^{s+k}}.
\]

When \( R(X) = X^\beta \) we use the simpler notation \( \zeta_{\mathcal{F}}(s, \beta) \).

- For each \( M \in \mathbb{R}_+ \) and each \( r \in \mathbb{Z} \), define the half-plane \( \mathcal{P}_M(r) \):

\[
\mathcal{P}_M(r) := \begin{cases} 
\{ s \in \mathbb{C} : \sigma > -M \} & \text{if (1) holds} \\
\{ s \in \mathbb{C} : \sigma > -M \} \cap \{ s \in \mathbb{C} : \sigma > e_{\mathcal{F}} + r \} & \text{if (2) holds.}
\end{cases}
\]

- For each \( \ell, r \in \mathbb{Z} \) and \( M \in \mathbb{R}_+ \), define the space of functions (of \( s \))

\[
\mathcal{E}_r(\ell, M) := \left\{ \sum_{k=1}^{n} P_k(s; c_1^{-s}, \ldots, c_d^{-s}) \zeta_{\mathcal{F}}(s, A_k) + \varphi(s) \right\},
\]

where \( d = \#f \) (see Definition 2) and

1. \( u < \infty \) and each \( A_k \in \mathcal{H} \) has degree at most \( \ell \);
2. \( \varphi(s) \) is holomorphic with moderate growth in \( \mathcal{P}_M(r) \);
3. each \( P_k \in \mathbb{C}[Y_0; Y_1, \ldots, Y_d] \).

Applying the linear transformation on spaces of monomials from Definition 5, the following two lemmas are keys to the proof of Theorem 2.

**Lemma 1.** Let \( r \in \mathbb{N}_0 \) and set \( N_r = \binom{n+r-1}{n-1} \). Denote by \( \alpha_1, \ldots, \alpha_N \), the elements \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n \) of weight \( r \) and ordered lexicographically.

For each \( j = 1, \ldots, d \), we define the real numbers \( g_{(u,v)}^{(j)} \) as in Definition 5. For each \( u \in \{1, \ldots, N_r\} \) set

\[
Z_u(s) := \zeta_{\mathcal{F}}(s, \alpha_u) - \sum_{v=1}^{N_r} \left( \sum_{j=1}^{d} g_{(u,v)}^{(j)} c_j^{-s} \right) \zeta_{\mathcal{F}}(s, \alpha_v).
\]

Then, for any \( M \in \mathbb{R}_+ \), \( Z_u(s) \in \mathcal{E}_r(r-1, M) \).

**Lemma 2.** Let \( r \geq 0 \) and \( \beta \) a vector of weight \( r \). Set \( R = \frac{X^\beta}{\| X \|^k} \in \mathcal{H} \) to be of degree \( \ell := r - k \). For \( \sigma > e_{\mathcal{F}} + \ell \) define (see (7)):

\[
V(s) := \delta_{\mathcal{F}}(s - \ell) \cdot \zeta_{\mathcal{F}}(s, R).
\]

Then, for any \( M \in \mathbb{R}_+ \), \( V \in \mathcal{E}_\ell(\ell - 1, M) \).

**Proof of Lemma 1:** Let \( \beta \in \{ \alpha_1, \ldots, \alpha_N \} \). By (5), we know that \( \zeta_{\mathcal{F}}(s, \beta) \) converges absolutely when \( \sigma > e_{\mathcal{F}} + r \).

We will fix once and for all \( M \in \mathbb{R}_+ \), and for two functions \( L, K \) define

\[
L \equiv K \quad \text{iff} \quad s \mapsto L(s) - K(s) \text{ has a holomorphic continuation with moderate growth to the half-plane } \mathcal{P}_M(r).
\]

With these notations, it again follows from (5) that for any \( i \neq i' \) and any \( \gamma \in \mathbb{N}_0^n \) such that \( |\gamma| \leq r \), the function \( s \mapsto \sum_{m \in f_i(\mathcal{F}) \cap f_{i'}(\mathcal{F})} \frac{m^\gamma}{\| m \|^s} \) converges absolutely and defines a holomorphic function with moderate growth in \( \mathcal{P}_M(r) \).
For each $j = 1, \ldots, d$ set
\[ x_j := -c_j^{-1}T_j^{-1}(b_j) = \text{the unique solution of the equation } f_j(x) = 0. \]

Using the bilinear form $\langle \cdot, \cdot \rangle$ associated to the norm $\| \cdot \|$ on $\mathbb{R}^n$, the following identity holds uniformly in $\sigma > e_F + r$:

\[
\zeta_F(s, \beta) := \sum_{m \in F'} \prod_{k=1}^n m_k^{\beta_k} \left\| \frac{1}{m} \right\|^s = \sum_{m \in F'} \prod_{k=1}^n \frac{\langle m, e_k \rangle^{\beta_k}}{\|m\|^s} = \sum_{j=1}^d \sum_{m \in F', m \not\cong x_j} \prod_{k=1}^n \frac{\langle f_j(m), e_k \rangle^{\beta_k}}{\|f_j(m)\|^s} = \sum_{j=1}^d \sum_{m \in F', m \not\cong x_j} \frac{\langle f_j(m), e_k \rangle^{\beta_k}}{\|m\|^s}.
\]

where $U_{j,\beta} \in \mathbb{C}[X]$ and $\deg(U_{j,\beta}) \leq r - 1$. Rearranging terms, we have uniformly in $\sigma > e_F + r$

\[
\zeta_F(s, \beta) = \sum_{j=1}^d \sum_{m \in F', m \not\cong x_j} \frac{c_j^s (T_j(m))^{\beta} + U_{j,\beta}(m)}{\|c_jT_j(m) + b_j\|^s} \left( \frac{\|c_jT_j(m) + b_j\|}{\|c_jT_j(m)\|} \right)^{-s}.
\]

where $m \in F'$ and $m \neq x_j$ imply $\|c_jT_j(m) + b_j\|^2 = \|c_jT_j(m)\|^2 \left[ 1 + \frac{K_j(m)}{\|m\|^2} \right]$, and $K_j$ is defined by the equation:

\[ K_j(m) := 2c_j^{-1}\langle T_j(m), b_j \rangle + c_j^{-2}\|b_j\|^2. \]

For each $j$ and any such $m$ we set $y_j(m) = K_j(m)/\|m\|^2$. It follows that $y_j(m) = O(1/\|m\|)$.

Defining now for each $N \geq 1$, $L_N(s, y) := (1 + y)^{-s} - \sum_{k=0}^N \left( \begin{smallmatrix} -s \\ k \end{smallmatrix} \right) y^k$, we then have:

\[
\zeta_F(s, \beta) = \sum_{j=1}^d \sum_{m \in F', m \not\cong x_j} \frac{\langle f_j(m), e_k \rangle^{\beta_k} + U_{j,\beta}(m)}{\|m\|^s} \left( \frac{\|c_jT_j(m) + b_j\|}{\|c_jT_j(m)\|} \right)^{-s}.
\]

In addition, by using (6) with $\beta = \alpha_u$, we deduce from (41):

For any $u = 1, \ldots, N_r$, the following holds uniformly in $\sigma > e_F + r$:

\[
\zeta_F(s, \alpha_u) = \sum_{k=0}^N \left( \begin{smallmatrix} -s/2 \\ k \end{smallmatrix} \right) \sum_{j=1}^d \sum_{m \in F', m \not\cong x_j} \frac{\langle f_j(m), \alpha_u \rangle^{\beta_k} + U_{j,\alpha_u}(m)}{\|m\|^s + 2k} \cdot L_N(s/2, y_j(m)).
\]
As a result, it is simple to check that \( \sigma > e_F + r \) implies that for each \( u = 1, \ldots, N_r \):

\[
Z_u(F; s) := \zeta_F(s, \alpha_u) - \sum_{v=1}^{N_r} \left( \sum_{j=1}^{d} g_{(u,v)}^{(j)} c_j^{r-s} \right) \zeta_F(s, \alpha_v)
\]

\[
= \sum_{k=1}^{N_r} \sum_{j=1}^{d} g_{(u,v)}^{(j)} \left( -\frac{s/2}{k} \right) c_j^{r-s} \zeta_F(s, K_{j,k,u}) + \sum_{k=0}^{N_r} \sum_{j=1}^{d} \left( -\frac{s/2}{k} \right) c_j^{r-s} \zeta_F(s, K_{j,k,u}) + \mathcal{L}_{N,u}(s),
\]

where

\[
K_{j,k,u}(X) = X^\alpha_u K_j(X)^k \quad \text{and} \quad \mathcal{U}_{j,k,u}(X) = \frac{U_{j,u}(X) K_j(X)^k}{\|X\|^{2k}},
\]

\[
\mathcal{L}_{N,u}(s) = \sum_{j=1}^{d} \sum_{m \in \mathcal{F}'} \frac{(c_j^{r-s} (T_j(m))^{\alpha_u} + c_j^{r-s} U_{j,u}(m))}{\|m\|^s} \cdot L_N(s/2, y_j(m)).
\]

It is clear that \( K_{j,k,u}, \mathcal{U}_{j,k,u} \in \mathcal{H} \) and:

\[
\deg K_{j,k,u}, \deg \mathcal{U}_{j,k,u} \leq r - k \leq r - 1 \quad \text{(since } k \geq 1); \quad \deg \mathcal{U}_{j,k,u} \leq r - 1 - k \leq r - 1.
\]

To finish the proof, it now suffices to choose \( N \) so that \( \mathcal{L}_{N,u} \) is holomorphic and has moderate growth in the halfplane \( \sigma > -M \). To this end, we set

\[
N = \lfloor e_F + M + r \rfloor + 1.
\]

Since \( y_j(m) = O(1/\|m\|) \), we may assume there exists \( \epsilon > 0 \) and an interval \([\delta, \gamma] \subset (-1, 1)\) such that

\[
y_j(m) \in [\delta, \gamma] \quad \text{for all } m \neq x_j \in \mathcal{F}' \cap \{\|m\| \geq b\}.
\]

We then apply the elementary estimate (see §2 [EL1]):

\[
|L_N(\sigma + it, y_j(m))| \ll_{b, \delta, \gamma, N, \sigma} (1 + |t|)^{N+1} |y_j(m)|^{N+1} \quad \text{uniformly in } t \text{ and } m \in \{\|m\| \geq b\}.
\]

This now gives the bound:

\[
\mathcal{L}_{N,u}(s) \ll_{b, \delta, \gamma, N, \sigma} (1 + |t|)^{N+1} \sum_{j=1}^{d} \frac{\|m\|^r}{\|m\|^s} \cdot |y_j(m)|^{N+1} \ll_{b, \delta, \gamma, N, \sigma} \sum_{m \in \mathcal{F}'} \frac{1}{\|m\|^{\sigma - r + N + 1}}.
\]

We then observe that \( \sigma - r + N + 1 > e_F + 1 \). Thus, the series on the second line converges absolutely in the halfplane \( \{\sigma > -M\} \) since the definition of \( e_F \) implies that \( \sum_{m \in \mathcal{F}'} \frac{1}{\|m\|^{e_F + r - \sigma}} \) converges.

It follows that \( \mathcal{L}_{N,u}(s) \) is both holomorphic and has moderate growth in \( \{\sigma > -M\} \). This completes the proof of Lemma 1. \( \diamond \)

**Proof of Lemma 2:**

Replacing \( s \) by \( s + k \), we reduce to the case in which \( \ell = r \geq 0 \) and \( R(X) = X^\beta \) with \( |\beta| = r \).

As in the proof of Lemma 1, \( \mathcal{M}_r(s) \) denotes the matrix in (7) and \( \delta_r(s) := \det (I_{N_r} - \mathcal{M}_r(s)) \).

The identities (40) imply that for \( \sigma > e_F + r \) we have:

\[
(I_{N_r} - \mathcal{M}_r(s-r)) \begin{pmatrix}
\zeta_F(s, \alpha_1) \\
\zeta_F(s, \alpha_2) \\
\vdots \\
\zeta_F(s, \alpha_{N_r})
\end{pmatrix} = \begin{pmatrix}
Z_1(F; s) \\
Z_2(F; s) \\
\vdots \\
Z_{N_r}(F; s)
\end{pmatrix}
\]
It follows that for \( \sigma > e_\mathcal{F} + r \) we have:

\[
\delta_r(s - r) \begin{pmatrix}
\zeta_\mathcal{F}(s, \alpha_1) \\
\zeta_\mathcal{F}(s, \alpha_2) \\
\vdots \\
\zeta_\mathcal{F}(s, \alpha_{N_r})
\end{pmatrix} = \text{Adj} \left( I_{N_r} - M_r(s - r) \right) \begin{pmatrix}
Z_1(\mathcal{F}; s) \\
Z_2(\mathcal{F}; s) \\
\vdots \\
Z_{N_r}(\mathcal{F}; s)
\end{pmatrix},
\]

where \( \text{Adj}(\cdot) \) denotes the adjugate matrix. Thus, Lemma 1 implies that for any \( u = 1, \ldots, N_r \) the function \( s \mapsto \delta_r(s - r) \zeta_\mathcal{F}(s, \alpha_u) \) belongs to \( \mathcal{E}_r(r - 1, M) \). This finishes the proof of Lemma 2. \( \diamond \)

To finish the proof of Theorem 2 we also need the following.

**Proposition 1.** Let \( M \in \mathbb{R}_+ \), and \( R \in \mathcal{H} \) be of degree \( \ell \in \mathbb{Z} \). Then, there exists a finite set \( S_0 = S_0(M, R) \subset \{(q, g) \in \mathbb{N}_0 \times \mathbb{Z} : g \leq q\} \) such that

\[
s \mapsto \left( \prod_{(q, g) \in S_0} \delta_q(s - g) \right) \zeta_\mathcal{F}(s, R)
\]

has a holomorphic continuation, with moderate growth, to the half-plane \( \mathcal{P}_M(\ell) \) (see (38)).

**Proof of Proposition 1:** The proof is by induction on \( \ell = \text{deg}(R) \). Throughout the discussion, we will work with a fixed and arbitrarily chosen \( M \in \mathbb{R}_+ \). We first set \( \ell_0 = [-M - e_\mathcal{F}] \).

**Step 1:** If \( \ell \leq \ell_0 \), then \( \sigma > -M \) implies \( \sigma > e_\mathcal{F} + \ell \). So, in this case, the proof follows from (5).

Thus, we may assume \( \ell \geq \ell_0 + 1 \).

**Step 2:** We assume the proposition holds for any \( R \in \mathcal{H} \) of degree at most \( \ell - 1 \).

**Step 3:** Given \( R \in \mathcal{H} \) of degree \( \ell \), it follows that there exist a finite set \( S(R) \subset \mathbb{N}_0^n \), an integer \( k \in \mathbb{N}_0 \), and complex numbers \( v_\beta \) such that

\[
R(X) = \sum_{\beta \in S(R)} v_\beta R_\beta(X) \quad (R_\beta = X^\beta / \| X \|^k),
\]

where \( \beta \in S(R) \) implies \( |\beta| - k \leq \ell \), and for some \( \beta \) there is equality. By (5), we know that \( \sigma > e_\mathcal{F} + \ell \) implies:

\[
\zeta_\mathcal{F}(s, R) = \sum_{\beta \in S(R)} v_\beta \zeta_\mathcal{F}(s, R_\beta).
\]

The induction hypothesis and Lemma 2 now imply that for any \( \beta \in S(R) \)

\[
\delta_{k + \ell}(s - \ell) \zeta_\mathcal{F}(s, R_\beta) \in \mathcal{E}_\ell(\ell - 1, M).
\]

Thus, (45) implies

\[
\delta_{k + \ell}(s - \ell) \zeta_\mathcal{F}(s, R) \in \mathcal{E}_\ell(\ell - 1, M).
\]

From the definition of \( \mathcal{E}_\ell(\ell - 1, M) \) and the induction hypothesis, it is clear that the elements of \( \mathcal{E}_\ell(\ell - 1, M) \) satisfy the conclusion of the Proposition. Iterating this argument finitely many times, the first step being that of replacing \( \ell \) by \( \ell - 1 \), then finishes the proof. \( \diamond \)

**Proof of Theorem 2:**

Proposition 1 implies that for any \( M \geq 0 \), there exists a finite set \( S_0(M) \subset \{(q, g) \in \mathbb{N}_0 \times \mathbb{Z} : g \leq q\} \) such that

\[
s \mapsto \left( \prod_{(q, g) \in S_0(M)} \delta_q(s - g) \right) \zeta_\mathcal{F}(s, \alpha)
\]
has a holomorphic continuation, with moderate growth, to the half-plane $\mathcal{P}_M(r)$ (see (38)) given that $r = |\alpha|$. By using in addition the following two points:

1. $\Delta_M(s) := \prod_{(q,g) \in S_0(M)} \delta_q(s-g)$ is an exponential polynomial of the form $\Delta_M(s) = 1 - \sum_{j=1}^{a} u_j e^{-w_j s}$ where $u_j \in \mathbb{R}$ and $w_j \in \mathbb{R}_+^*$, $\forall j = 1, \ldots, a$;

2. Denoting by $V$ the set of complex zeros of $\Delta_M(s)$ that belong to a given vertical strip of finite width, then, for any $\theta > 0$, there exists $\mu(\theta) > 0$ such that $d(s, V) \geq \theta$ implies $|\Delta_M(s)| \geq \mu(\theta)$ (see ([L], Lemma 1 pg. 268),

we deduce that for any $M \in \mathbb{R}_+$, the zeta function $\zeta_\mathcal{F}(s, \alpha)$ has a meromorphic continuation with moderate growth to the half-plane $\mathcal{P}_M(r)$ with poles located in the set

$$\bigcup_{(q,g) \in S_0(M)} \{s + g : \delta_q(s) = 0\} \subset \bigcup_{q \in \mathbb{N}_0} \bigcup_{k \leq q} \{s + k : \delta_q(s) = 0\}.$$

By letting $M \to \infty$, this proves Parts 1, 2, and 3(a) of Theorem 2.

**Proof of 3(b):**

Fix $M_0 \in \mathbb{R}_+$ such that $M_0 > -r - e_\mathcal{F}^r$, and set $V(s) := \delta_r(s-r) \cdot \zeta_\mathcal{F}(s, \alpha)$.

We then use Lemma 2 with $R = x^\alpha$. Thus, $\ell = r$, and the Lemma implies that $V \in \mathcal{E}_r(r-1, M_0)$. It follows that there exist finitely many $A_1, \ldots, A_u$ of $\mathcal{H}$ of degree at most $r-1$, finitely many polynomials $P_k(s, Y_1, \ldots, Y_u)$ ($1 \leq k \leq u$), and a function $\varphi$ holomorphic with moderate growth in $\mathcal{P}_{M_0}(r)$ such that

$$V(s) = \sum_{k=1}^{u} P_k(s; c_1^{-s}, \ldots, c_u^{-s}) \zeta_\mathcal{F}(s, A_k) + \varphi(s).$$

By (5), we know that for each $k = 1, \ldots, u$, $\zeta_\mathcal{F}(s, A_k)$ converges absolutely when $\sigma > e_\mathcal{F} + r - 1$.

The proof of 3(b) now follows by using the fact that $\{\sigma > e_\mathcal{F} + r\} \subset \mathcal{P}_{M_0}(r)$. This finishes the proof of Theorem 2.

### 3.2 Proof of Theorem 1

Parts 2, 3, 4(a) and 4(b) of Theorem 1 follow immediately from Theorem 2, so it suffices to prove parts 1 and 4(c).

**Proof of (1):** Define the function $g : \mathbb{R} \to \mathbb{R}$ by $g(\sigma) = \sum_{i=1}^{d} c_i^{-\sigma}$, It is easy to see that $g$ is a monotone decreasing function that satisfies $\lim_{\sigma \to \infty} g(\sigma) = 0$ and $\lim_{\sigma \to -\infty} g(\sigma) = \infty$.

Thus, there exists a unique $\sigma_0 \in \mathbb{R}$ such that $g(\sigma_0) = 1$. It follows that $\sigma_0$ is the unique real solution of the Dirichlet polynomial $\delta_0(s) = 1 - \sum_{j=1}^{d} c_j^{-s}$.

We now show that $\sigma_0 = e_\mathcal{F}$:

A standard result (see [HR]) is that $e_\mathcal{F}$ is the abscissa of convergence of $\zeta_\mathcal{F}(s)$. Moreover, Part 4(b) implies $\eta > 0$ exists such that $s \mapsto \delta_0(s) \cdot \zeta_\mathcal{F}(s)$ is holomorphic in the half-plane $\{\sigma > e_\mathcal{F} - \eta\}$. Landau’s theorem [ibid.] now tells us that $e_\mathcal{F}$ is a singular point of $\zeta_\mathcal{F}(s)$.
Thus, \( e_\mathcal{F} \) is necessarily a real zero of \( \delta_0(s) \). This implies that \( e_\mathcal{F} = \sigma_0 \), which finishes the proof of Part 1.

- **Proof of 4(c):** The polar order of \( \zeta_\mathcal{F} \) at \( s = e_\mathcal{F} \) is at most equal to the multiplicity of \( e_\mathcal{F} \) as a zero of \( \delta_0(s) \). Since \( \delta'_0(e_\mathcal{F}) = \sum_{j=1}^d \log c_j c_j^{-e_\mathcal{F}} \neq 0 \), it follows that this pole must be simple. This finishes the proof of Theorem 1. \( \diamond \)

# 4 Proofs of multivariate Tauberian Theorems 3 and 4

## 4.1 Four Lemmas

We will need four preliminaries before proving Theorems 3 and 4 in §4.2, §4.3. The proofs of the Lemmas are given in §4.4. The first is a consequence of Ingham’s theorem for Dirichlet series with positive coefficients.

**Lemma 3.** Let \( Z(a, s) = \sum_{n=1}^\infty \frac{a_n}{\lambda_n^s} \) be a Dirichlet series such that \( a_n \geq 0 \ \forall n \). Assume there exists \( \mu, A_0 \in \mathbb{R} \) and \( \rho_0 \in \mathbb{N}_0 \) such that \( Z(a, s) \) converges absolutely for \( \sigma > \mu \) and

\[
Z(a, \sigma) = \frac{A_0}{(\sigma - \mu)^{\rho_0}} + O \left( \frac{1}{(\sigma - \mu)^{\rho_0-1}} \right) \quad \text{as} \ \sigma \to \mu + .
\]

Then,

\[
S_\mu(a, t) := \sum_{\lambda_n \leq t} \frac{a_n}{\lambda_n^s} = \frac{A_0}{\rho_0!} (\log t)^{\rho_0} + O \left( \frac{(\log t)^{\rho_0}}{\log(\log t)} \right) \quad \text{as} \ t \to \infty.
\]

The second Lemma is an extension to complex poles:

**Lemma 4.** Let \( Z(a, s) = \sum_{n=1}^\infty \frac{a_n}{\lambda_n^s} \) and \( Z(b, s) = \sum_{n=1}^\infty \frac{b_n}{\lambda_n^s} \) be two Dirichlet series such that \( |b_n| \leq a_n \) for each \( n \). Denote by \( \sigma_a \) the abscissa of convergence of \( Z(a, s) \). Assume \( Z(a, s) \) and \( Z(b, s) \) have meromorphic extensions to an open set containing \( \{ s \in \mathbb{C} : \sigma \geq \sigma_a \} \). Set \( \rho_0 = \text{ord}_{s=\sigma_a} Z(a, s) \).

If \( s_0 \) is a pole of \( Z(b, s) \) on the line \( \sigma = \sigma_a \), then \( \text{ord}_{s=s_0} Z(b, s) \leq \rho_0 \).

The third Lemma is a type of hybrid of Lemmas 3, 4.

**Lemma 5.** Let \( Z(a, s) = \sum_{n=1}^\infty \frac{a_n}{\lambda_n^s} \) and \( Z(b, s) = \sum_{n=1}^\infty \frac{b_n}{\lambda_n^s} \) be two Dirichlet series with real coefficients such that \( |b_n| \leq a_n \) for each \( n \). Denote by \( \sigma_a \) the abscissa of convergence of \( Z(a, s) \), and assume that \( Z(a, s) \) and \( Z(b, s) \) have meromorphic continuations to an open set containing \( \{ s \in \mathbb{C} : \sigma \geq \sigma_a \} \).

We set \( \rho_0 = \text{ord}_{s=\sigma_a} Z(a, s) \) and \( B_0 := \lim_{s \to \sigma_a} (s - \sigma_a)^{\rho_0} Z(b, s) \).

Then \( B_0 \) exists, is finite, and the average

\[
S_{\sigma_a}(b, t) := \sum_{\lambda_n \leq t} \frac{b_n}{\lambda_n^{\sigma_a}} = \frac{B_0}{\rho_0!} (\log t)^{\rho_0} + O \left( \frac{(\log t)^{\rho_0}}{\log(\log t)} \right) \quad \text{as} \ t \to \infty.
\]
The fourth Lemma is purely combinatorial.

**Lemma 6.** Let $F$ be a discrete subset of $\mathbb{R}^n$. Let $\psi : F^h \to \mathbb{C}$ be a function. Let $x_1, \ldots, x_h$ and $y_1, \ldots, y_h$ be real numbers such that $y_i < x_i \; \forall i = 1, \ldots, h$. Then

$$\sum_{\substack{m_1, \ldots, m_h \in F \\ y_j < \|m_j\| \leq x_j \; \forall j = 1, \ldots, h}} \psi(m_1, \ldots, m_h) = \sum_{I \subset \{1, \ldots, h\}} (-1)^{|I|} \sum_{\substack{m_1, \ldots, m_h \in F \\ \|m_j\| \leq y_j \; \forall j \in I \text{ and } \|m_j\| \leq x_j \; \forall j \not\in I}} \psi(m_1, \ldots, m_h).$$

4.2 Proof of Theorem 4

Denote by $\xi_1 < \xi_2 < \ldots$ the increasing sequence formed by the elements $\|m\|$ $(m \in F')$. Let $\alpha = (\alpha_1, \ldots, \alpha_k) \in \text{Supp} (R, v)$ (see (16)). Defining the sequences

$$b_n(\alpha_j) := \sum_{\|m\| = \xi_n} m^{\alpha_j} \quad \text{and} \quad a_n := \sum_{\|m\| = \xi_n} \|m\|^v_j = \# \{m \in F' : \|m\| = \xi_n \} \cdot \xi_n^v_j,$$

it is clear that if $\sigma$ is sufficiently large then

$$Z(b(\alpha_j), s) := \sum_{n=1}^{\infty} b_n(\alpha_j) \xi_n^s = \sum_{m \in F'} \frac{m^{\alpha_j}}{\|m\|^s} = \zeta_F(s, \alpha_j)$$

and

$$Z(a, s) := \sum_{n=1}^{\infty} a_n \xi_n^s = \sum_{m \in F'} \frac{\|m\|^v_j}{\|m\|^s} = \zeta_F(s - v_j).$$

Recalling Part 4(c) of Theorem 1, we note that

$$\text{ord}_{s = e_F + v_j} \zeta_F(s - v_j) = \text{ord}_{s = e_F} \zeta_F(s) = 1.$$

Furthermore, since $|m^{\alpha_j}| \leq \|m\|^v_j$, Lemma 4 implies that if $\zeta_F(s, \alpha_j)$ has a pole at $e_F + v_j$, then it must be simple.

By applying Lemma 5 to $Z(a, s)$ and $Z(b(\alpha_j), s)$, we deduce that for any $\alpha = (\alpha_1, \ldots, \alpha_k) \in \text{Supp} (R, v)$ and each $j \in \{1, \ldots, k\}$:

$$\sum_{0 < \|m\| \leq x_j} \frac{m^{\alpha_j}}{\|m\|^v_j} = \text{Res}_{s_j = e_F + v_j} \zeta_F(s_j, \alpha_j) \cdot \log x_j + O \left( \frac{\log x_j}{\log(\log x_j)} \right) \text{ as } x_j \to \infty. \quad (46)$$

**Notation.** For the rest of the proof, it is convenient to set

$$d := e_F 1_k + v = (d_1, \ldots, d_k). \quad (47)$$

For simplicity, we write $d$ instead of $d_v$. \qed
Applying Lemma 6, we deduce that for each $j$ and all sufficiently large $x_j$ we have

$$A_d(R_v, x_1, \ldots, x_k)$$

$$= \sum_{\alpha \in \text{Supp} (R_v)} b(\alpha) \prod_{j=1}^{k} \left( \sum_{0 \leq m_i \leq x_j} \frac{m^{\alpha_j}}{||m||^{v_j+\delta_j}} \right)$$

$$= \sum_{\alpha \in \text{Supp} (R_v)} b(\alpha) \prod_{j=1}^{k} \left( \text{Res}_{s_j=d_j} \zeta_F(s_j, \alpha_j) \cdot \log x_j + O\left( \frac{\log x_j}{\log(\log x_j)} \right) \right)$$

$$= \left( \sum_{\alpha \in \text{Supp} (R_v)} b(\alpha) \prod_{j=1}^{k} \text{Res}_{s_j=d_j} \zeta_F(s_j, \alpha_j) \right) \cdot (\log x_1 \ldots \log x_k) \cdot \left( 1 + O\left( \sum_{j=1}^{k} \frac{1}{\log(\log x_j)} \right) \right).$$

This finishes the proof of (19).

**Proof of (20):**

Set $\mu \in \prod_{j=1}^{k} [0, d_j)$ and let $\delta \in (0, 1)$ be arbitrary. Then, we have for each $j$ and all sufficiently large $x_j$:

$$A_\mu(R_v, x_1, \ldots, x_k) = \sum_{m_1, \ldots, m_k \in F} R_v(m_1, \ldots, m_k) \gtrsim \sum_{m_1, \ldots, m_k \in F} R_v(m_1, \ldots, m_k) \left( \prod_{j=1}^{k} \frac{d_j^{\delta(d_j-\mu_j)}}{d_j} \right) \cdot \sum_{m_1, \ldots, m_k \in F} R_v(m_1, \ldots, m_k) \left( \prod_{j=1}^{k} \frac{d_j^{\delta(d_j-\mu_j)}}{d_j} \right).$$

For any $I \subset \{1, \ldots, k\}$, we define $f_{I, \delta, i}(x) = x_i^\delta$ if $i \in I$ and $f_{I, \delta, i}(x) = x_i$ if $i \notin I$.

Applying Lemma 6, we deduce that for $\delta \in (0, 1)$, for each $j$ and all sufficiently large $x_j$:

$$A_\mu(R_v, x_1, \ldots, x_k) \gtrsim \left( \prod_{j=1}^{k} x_j^{\delta(d_j-\mu_j)} \right) \left[ \sum_{I \subset \{1, \ldots, k\}} (-1)^{#I} A_d(R_v, f_{I, \delta, 1}(x), \ldots, f_{I, \delta, k}(x)) \right].$$

Moreover, (19) implies that

$$H_\delta(x) = \sum_{I \subset \{1, \ldots, k\}} (-1)^{#I} A_d(R_v, f_{I, \delta, 1}(x), \ldots, f_{I, \delta, k}(x))$$

$$= A_0(R_v) \cdot \left( \sum_{I \subset \{1, \ldots, k\}} (-1)^{#I} \delta^{#I} \right) \left( \log x_1 \ldots \log x_k \right) + O_\delta \left( \sum_{j=1}^{k} \frac{\log x_1 \ldots \log x_k}{\log(\log x_j)} \right)$$

$$= A_0(R_v) \cdot (1 - \delta)^k \left( \log x_1 \ldots \log x_k \right) + O_\delta \left( \sum_{j=1}^{k} \frac{\log x_1 \ldots \log x_k}{\log(\log x_j)} \right).$$
Combining (48), (49), and using the fact that \( A_0(R_\nu) \cdot (1 - \delta)^k > 0 \), we deduce that
\[
\forall \delta \in (0, 1), \quad A_\mu(R_\nu, x_1, \ldots, x_k) \gg \delta \prod_{j=1}^k x_j^{\delta(d_j - \mu_j)} \quad \text{as} \quad \inf_{1 \leq j \leq k} x_j \to \infty.
\]
Applying (47), this finishes the proof of (20) and Theorem 4. \(\diamond\)

4.3 Proof of Theorem 3

Let \( R(X_1, \ldots, X_k) = \sum_{\alpha \in \text{Supp}(R)} b(\alpha) X^\alpha = \sum_{\alpha \in \text{Supp}(R)} b(\alpha) X_1^{\alpha_1} \cdots X_k^{\alpha_k} \) satisfy the hypotheses 1, 2 of Theorem 3. Define the polynomial \( H \) by
\[
H(X_1, \ldots, X_k) = \sum_{\alpha \in \text{Supp}(R) \setminus \text{Supp}_0(R)} b(\alpha) X_1^{\alpha_1} \cdots X_k^{\alpha_k}. \quad (50)
\]
It follows that for any \( x = (x_1, \ldots, x_k) \in (0, \infty)^k \)
\[
\mathcal{A}(R, x_1, \ldots, x_k) := \sum_{v \in E(R)} \mathcal{A}(R_v, x_1, \ldots, x_k) + \mathcal{A}(H, x_1, \ldots, x_k). \quad (51)
\]
Set
\[
\omega := \sup \{|\alpha| : \alpha \in \text{Supp}(R) \setminus \text{Supp}_0(R)\}; \quad \varepsilon_0 = \frac{\text{deg}(R) - \omega}{1 + \omega} > 0; \quad \varepsilon \in (0, \varepsilon_0). \quad (52)
\]
It is clear that for any such \( \varepsilon \), there exists \( \theta(= \theta(\varepsilon)) = (\theta_1, \ldots, \theta_k) \in [1, 1 + \varepsilon]^k \) so that
\[
\langle v, \theta \rangle \neq \langle v', \theta \rangle \quad \forall v \neq v' \in \mathcal{V}(R).
\]
It follows that there exists \( v_0 (= v_0(\theta)) \in \mathcal{V}(R) \) such that
\[
\langle v_0, \theta \rangle > \langle v, \theta \rangle \quad \forall v \in \mathcal{V}(R) \setminus \{v_0\}. \quad (53)
\]
Defining \( \mathcal{E}_{v_0}(x_1, \ldots, x_k) = \sum_{v \in E(R) \setminus \{v_0\}} \mathcal{A}(R_v, x_1, \ldots, x_k) \), it is then clear that
\[
\mathcal{A}(R, x_1, \ldots, x_k) := \mathcal{A}(R_{v_0}, x_1, \ldots, x_k) + \mathcal{E}_{v_0}(x_1, \ldots, x_k) + \mathcal{A}(H, x_1, \ldots, x_k). \quad (54)
\]
Claim I: Set \( \omega_1 = \omega_1(\theta) := \sup \{|v, \theta| : v \in E(R) \setminus \{v_0\}\} \). Then
1. \( \omega_1 < \langle v_0, \theta \rangle \);
2. \( \forall \eta_1 > 0, \quad \mathcal{E}_{v_0}(x^{\theta_1}, \ldots, x^{\theta_k}) \ll \eta_1 \varepsilon x^{\omega_1 + |\theta| (\varepsilon_0 + \eta_1)} \quad \text{as} \quad x \to \infty. \)

Proof of (1): Let \( v \in E(R) \setminus \{v_0\} \). From the definition (15) of \( \mathcal{V}(R) \) it follows that there exist \( v_1, \ldots, v_q \in \mathcal{V}(R) \setminus \{v_0\} \) and \( \lambda_0, \ldots, \lambda_q \in [0, 1] \) satisfying \( \lambda_0 + \cdots + \lambda_q = 1 \) such that
\[
v = \sum_{i=0}^q \lambda_i v_i.
\]
The assumption $v \neq v_0$ implies that there exists at least one $i \in \{1, \ldots, q\}$ such that $\lambda_i > 0$. It follows then from (53) that

$$\langle v, \theta \rangle = \lambda_0 \langle v_0, \theta \rangle + \sum_{i=1}^{q} \lambda_i \langle v_i, \theta \rangle < \lambda_0 \langle v_0, \theta \rangle + \sum_{i=1}^{q} \lambda_i \langle v_0, \theta \rangle = \langle v_0, \theta \rangle.$$ 

This proves (1).

**Proof of (2):** From Definition 3 it follows that for any $v \in E(R) \setminus \{v_0\}$ and $\eta_1 > 0$:

$$|A(R_v, x^{\theta_1}, \ldots, x^{\theta_k})| = \left| \sum_{\alpha \in \text{Supp}(R,v)} b(\alpha) \sum_{\substack{m_1, \ldots, m_k \in F^t \\\|m_j\| \leq x^{\theta_j} \text{ for all } j = 1, \ldots, k}} m_1^{\alpha_1} \ldots m_k^{\alpha_k} \right| \leq \sum_{\alpha \in \text{Supp}(R,v)} b(\alpha) \sum_{\substack{m_1, \ldots, m_k \in F^t \\\|m_j\| \leq x^{\theta_j} \text{ for all } j = 1, \ldots, k}} \|m_1\|^{\alpha_1} \ldots \|m_k\|^{\alpha_k} \ll x^{\langle v, \theta \rangle} \prod_{j=1}^{k} \left( \sum_{\{m_j \in F^t \|m_j\| \leq x^{\theta_j}\}} 1 \right) \ll_{\eta_1, \varepsilon} x^{\langle v, \theta \rangle + \|\varepsilon x + \eta_1\|} = x^{\langle v, \theta \rangle + \|\varepsilon x + \eta_1\|} \ll_{\eta_1, \varepsilon} x^{\omega_1 + \|\varepsilon x + \eta_1\|} \quad \text{as } x \to \infty.
$$

Since $E_v(\cdot) = \sum_{v \in E(R) \setminus \{v_0\}} A(R_v, \cdot)$, this proves (2). \[\square\]

**Claim II:** For any $\eta_2 > 0$,

$$A(H, x^{\theta_1}, \ldots, x^{\theta_k}) \ll_{\eta_2, \varepsilon} x^{(1+\varepsilon)\omega + \|\varepsilon x + \eta_2\|} \quad \text{as } x \to \infty. \quad (55)$$

**Proof:** As with Claim 1, we have

$$|A(H, x^{\theta_1}, \ldots, x^{\theta_k})| = \left| \sum_{\alpha \in \text{Supp}(R) \setminus \text{Supp}(v_0)} b(\alpha) \sum_{\substack{m_1, \ldots, m_k \in F^t \\\|m_j\| \leq x^{\theta_j} \text{ for all } j = 1, \ldots, k}} m_1^{\alpha_1} \ldots m_k^{\alpha_k} \right| \ll \sum_{\alpha \in \text{Supp}(R) \setminus \text{Supp}(v_0)} b(\alpha) \sum_{\substack{m_1, \ldots, m_k \in F^t \\\|m_j\| \leq x^{\theta_j} \text{ for all } j = 1, \ldots, k}} \|m_1\|^{\alpha_1} \ldots \|m_k\|^{\alpha_k} \ll_{\sum_{j=1}^{k} \theta_j |\alpha_j|} \prod_{j=1}^{k} \left( \sum_{\{m_j \in F^t \|m_j\| \leq x^{\theta_j}\}} 1 \right) \ll_{\eta_2, \varepsilon} x^{\sum_{j=1}^{k} \theta_j |\alpha_j| + \|\varepsilon x + \eta_2\|} \ll_{\eta_2, \varepsilon} x^{(1+\varepsilon)\omega + \|\varepsilon x + \eta_2\|} \quad \text{as } x \to \infty.$$

This proves Claim II.

**Finishing the proof of Theorem 3:** We first observe that the lower bound (20) of Theorem 4 implies that for any $\kappa > 0$:

$$A(R_{v_0}, x^{\theta_1}, \ldots, x^{\theta_k}) \gg_{\kappa, \varepsilon} x^{\langle v_0, \theta \rangle + \varepsilon|x| - \kappa} \quad \text{as } x \to \infty. \quad (56)$$

Writing

$$A(R, x^{\theta_1}, \ldots, x^{\theta_k}) = A(R_{v_0}, x^{\theta_1}, \ldots, x^{\theta_k}) \cdot \left(1 + \frac{E_v(x^{\theta_1}, \ldots, x^{\theta_k})}{A(R_{v_0}, x^{\theta_1}, \ldots, x^{\theta_k})} + \frac{A(H, x^{\theta_1}, \ldots, x^{\theta_k})}{A(R_{v_0}, x^{\theta_1}, \ldots, x^{\theta_k})} \right), \quad (57)$$
we next show that for large $x$ and any $\varepsilon \in (0, \varepsilon_0)$, the left most factor can be bounded away from 0, and each of the two quotients on the right side can be made arbitrarily small by suitable choices of $\eta_1, \eta_2, \kappa$. To do so, we will use (56) three times. It is then important to note that this entails three a priori independent choices for the parameter $\kappa$, two of which are needed to make the two quotients small, while the third is used to bound the leftmost factor away from 0.

From the definition of $\omega_1$, Part 2 of Claim I, and (56) it follows that

$$
\frac{\mathcal{E}_{\nu_0}(x^{\theta_1}, \ldots, x^{\theta_k})}{\mathcal{A}(\mathcal{R}_{\nu_0}, x^{\theta_1}, \ldots, x^{\theta_k})} \ll_{\eta_1, \kappa, \varepsilon} x^{(1+\varepsilon)\omega-(\nu_0, \theta)+|\theta|\eta_1+\kappa_1}.
$$

(58)

Since $\omega_1 - \langle \nu_0, \theta \rangle < 0$ and $|\theta| \leq k(1 + \varepsilon_0)$ for any $\varepsilon \in (0, \varepsilon_0)$, we can always choose $\eta_1 = \eta_1(\varepsilon) > 0$ and $\kappa_1 = \kappa_1(\varepsilon) > 0$ so that $|\theta|\eta_1 + \kappa_1 < \frac{1}{2} \cdot (\langle \nu_0, \theta \rangle - \omega_1)$. Thus:

$$
\nu_1 = \nu_1(\eta_1, \kappa_1, \varepsilon) := \omega_1 - \langle \nu_0, \theta \rangle + |\theta|\eta_1 + \kappa_1 \leq \frac{1}{2} (\omega_1 - \langle \nu_0, \theta \rangle) < 0 \quad \forall \varepsilon \in (0, \varepsilon_0).
$$

(59)

Moreover, since $\langle \nu_0, \theta \rangle \geq |\nu_0| = \deg R$, it follows from Claim II and (56) that for any $\varepsilon \in (0, \varepsilon_0)$:

$$
\frac{\mathcal{A}(H, x^{\theta_1}, \ldots, x^{\theta_k})}{\mathcal{A}(\mathcal{R}_{\nu_0}, x^{\theta_1}, \ldots, x^{\theta_k})} \ll_{\eta_2, \kappa, \varepsilon} x^{(1+\varepsilon)\omega-(\nu_0, \theta)+|\theta|\eta_2+\kappa_2} \ll_{\eta_2, \kappa, \varepsilon} x^{-(1+\varepsilon)\omega-\deg R+|\theta|\eta_2+\kappa_2} \quad \text{as } x \to \infty.
$$

(60)

Since $\omega - \deg R = -(1 + \varepsilon)\varepsilon_0$ and $|\theta| \leq k(1 + \varepsilon_0)$, we have

$$(1+\varepsilon_0)\omega - \deg R + |\theta|\eta_2 + \kappa_2 \leq -\varepsilon_0 + k(1 + \varepsilon_0)\eta_2 + \kappa_2,$$

and this upper bound is independent of $\varepsilon$ and strictly negative by (52) if $\eta_2$ and $\kappa_2$ are small enough. Thus we conclude that $\eta_2, \kappa_2 > 0$ can be chosen so that

$$
\nu_2(\eta_2, \kappa_2) := (1+\varepsilon_0)\omega - \deg R + |\theta|\eta_2 + \kappa_2 < 0.
$$

(61)

In addition, having chosen $\eta_1, \eta_2, \kappa_1, \kappa_2$ so that (59) and (61) hold, (57) now implies the existence of $c_1 = c_1(\kappa_1, \varepsilon), c_2 = c_2(\kappa_2) > 0$ such that:

$$
\mathcal{A}(R, x^{\theta_1}, \ldots, x^{\theta_k}) = \mathcal{A}(\mathcal{R}_{\nu_0}, x^{\theta_1}, \ldots, x^{\theta_k}) \cdot \left(1 + \frac{\mathcal{E}_{\nu_0}(x^{\theta_1}, \ldots, x^{\theta_k})}{\mathcal{A}(\mathcal{R}_{\nu_0}, x^{\theta_1}, \ldots, x^{\theta_k})} + \frac{\mathcal{A}(H, x^{\theta_1}, \ldots, x^{\theta_k})}{\mathcal{A}(\mathcal{R}_{\nu_0}, x^{\theta_1}, \ldots, x^{\theta_k})}\right) \gg_{\kappa_3, \varepsilon} x^{(\nu_0, \theta)+|\theta|-\kappa_3(1-c_1x^{\nu_1}-c_2x^{\nu_2})} \gg_{\kappa_3, \varepsilon} x^{\deg R+ke_{\varepsilon}-\kappa_3} \quad \text{as } x \to \infty,
$$

(62)

where it is understood that the rightmost exponent of $x$ in (62) depends only upon $\kappa_3$ and that the implied constant is positive and depends upon $\varepsilon$ and $\kappa_3$.

We next use the fact that $\mathcal{A}(R, x_1, \ldots, x_k)$ is increasing in each variable $x_j$. Denoting the parameter $\kappa_3$ by $\kappa$ we now conclude that for any sufficiently small $\varepsilon > 0$ and $\theta \in [1, 1 + \varepsilon]^k$,

$$
\mathcal{A}(R, x, \ldots, x) \geq \mathcal{A} \left( R; \left(\frac{x}{1+\varepsilon}\right)^{\theta_1}, \ldots, \left(\frac{x}{1+\varepsilon}\right)^{\theta_k} \right) \gg_{\varepsilon, \kappa} x^{\frac{\deg R+ke_{\varepsilon}-\kappa}{1+\varepsilon}} \gg_{\delta} x^{\deg R+ke_{\varepsilon}-\delta} \quad \text{as } x \to \infty,
$$

where the rightmost bound means that given any sufficiently small $\delta > 0$, there exist $\kappa$ and $\varepsilon$ so that $\frac{\deg R+ke_{\varepsilon}-\kappa}{1+\varepsilon} > \deg R + ke_{\varepsilon} - \delta$. This completes the proof of Theorem 3. ◇
4.4 Proof of Lemmas 3, 4, 5 and 6

Proof of Lemma 3:
For each $n$, we define $c_n = \frac{a_n}{\lambda_n^s}$, and $Z^*(s) := \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n^s} = Z(a; s + \mu)$. Set $S^*(t) := \sum_{\lambda_n \leq t} c_n$.

First we remark that for $\sigma > 0$, $F(s) := \int_0^\infty e^{-sx} S^*(e^x) \, dx = \frac{Z^*(s)}{s}$.

It follows that $F(s)$ converges for $\sigma > 0$ and satisfies

$$ F(\sigma) = \frac{A_0}{\sigma^{\rho_0+1}} + O \left( \frac{1}{\sigma^{\rho_0}} \right) \quad \text{as } \sigma \to 0^+. $$

By using Ingham’s theorem (Theorem B of [I]) with $\chi(u) = u$, we deduce that

$$ S^*(e^x) = \frac{A_0}{\rho_0!} x^{\rho_0} + O \left( \frac{1}{\log x} \right) \quad \text{as } x \to \infty. $$

It follows that

$$ S^*(t) = \sum_{\lambda_n \leq t} \frac{a_n}{\lambda_n^s} = \frac{A_0}{\rho_0!} (\log t)^{\rho_0} + O \left( \frac{1}{\log(\log t)} \right) \quad \text{as } t \to \infty. $$

This finishes the proof of Lemma 3.

Proof of Lemma 4: Assume that $s_0 = \sigma_a + it_0$ is a pole of $Z(b; s)$ on the vertical line $\sigma = \sigma_a$. Denote by $\rho$ the order of $s_0$ as a pole of $Z(b; s)$. It follows that there exists a constant $B \neq 0$ such that

$$ Z(b; \sigma + it_0) \sim B (\sigma - \sigma_a)^{-\rho} \quad \text{as } \sigma \to \sigma_a \quad (\sigma > \sigma_a). \quad (63) $$

Similarly, since $\sigma_a$ is a pole of order $\rho_0$, there exists $A_0 \neq 0$ such that

$$ Z(a; \sigma) \sim A_0 (\sigma - \sigma_a)^{-\rho_0} \quad \text{as } \sigma \to \sigma_a \quad (\sigma > \sigma_a). \quad (64) $$

It is then clear that for any $\sigma \in (\sigma_a, \infty)$,

$$ |Z(b; \sigma + it_0)| = \left| \sum_{n=1}^{\infty} \frac{b_n}{\lambda_n^{\sigma + it_0}} \right| \leq \sum_{n=1}^{\infty} \frac{|b_n|}{\lambda_n^{\sigma}} \leq \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^{\sigma}} = Z(a; \sigma). \quad (65) $$

We therefore see that (63) - (65) imply $(\sigma - \sigma_a)^{-\rho} \ll (\sigma - \sigma_a)^{-\rho_0}$ as $\sigma \to \sigma_a \quad (\sigma > \sigma_a)$. Noting this can only occur when $\rho \leq \rho_0$ then finishes the proof.

Proof of Lemma 5: Let $\rho \geq 0$ be the order of $\sigma_a$ as a (possible) pole of $Z(b; s)$. Lemma 4 implies that $\rho \leq \rho_0$. It follows that $B_0 := \lim_{s \to \sigma_a} (s - \sigma_a)^{\rho_0} Z(b; s)$ is finite (possibly 0) and that

$$ Z(b; \sigma) = \frac{B_0}{(\sigma - \sigma_a)^{\rho_0}} + O \left( \frac{1}{(\sigma - \sigma_a)^{\rho_0-1}} \right) \quad \text{as } \sigma \to \sigma_a. \quad (66) $$
We further know that

\[ Z(a; \sigma) = \frac{A_0}{(\sigma - \sigma_a)^{\rho_0}} + O\left(\frac{1}{(\sigma - \sigma_a)^{\rho_0-1}}\right) \quad \text{as } \sigma \to \sigma_a, \quad (67) \]

where \( A_0 := \lim_{s \to \sigma_a} (s - \sigma_a)^{\rho_0} Z(a; s) \).

Define the sequence \( d = (d_n)_{n \geq 1} \) by \( d_n = a_n - b_n \) for any \( n \geq 1 \).

Then, \( d_n \geq 0 \) for any \( n \geq 1 \), \( Z(d; s) := \sum_{n=1}^{\infty} \frac{d_n}{\lambda_n^s} \) converges absolutely in \( \{\sigma > \sigma_a\} \) and

\[ Z(d; \sigma) = \frac{A_0 - B_0}{(\sigma - \sigma_a)^{\rho_0}} + O\left(\frac{1}{(\sigma - \sigma_a)^{\rho_0-1}}\right) \quad \text{as } \sigma \to \sigma_a. \]

Applying Lemma 3 gives both

\[ S_{\sigma_a}(d; t) := \sum_{\lambda_n \leq t} \frac{d_n}{\lambda_n^{\sigma_a}} = \frac{(A_0 - B_0)}{\rho_0!} (\log t)^{\rho_0} + O\left(\frac{(\log t)^{\rho_0}}{\log(\log t)}\right) \quad \text{as } t \to \infty. \quad (68) \]

and

\[ S_{\sigma_a}(a; t) := \sum_{\lambda_n \leq t} \frac{a_n}{\lambda_n^{\sigma_a}} = \frac{A_0}{\rho_0!} (\log t)^{\rho_0} + O\left(\frac{(\log t)^{\rho_0}}{\log(\log t)}\right) \quad \text{as } t \to \infty. \quad (69) \]

Combining (68) and (69) now implies

\[ S_{\sigma_a}(b, t) = S_{\sigma_a}(a; t) - S_{\sigma_a}(d; t) = \frac{B_0}{\rho_0!} (\log t)^{\rho_0} + O\left(\frac{(\log t)^{\rho_0}}{\log(\log t)}\right) \quad \text{as } t \to \infty, \]

which finishes the proof of Lemma 5. ♦

**Proof of Lemma 6:** We argue by induction on \( h \in \mathbb{N} \).

• If \( h = 1 \), the lemma reduce to the formula

\[ \sum_{m_1 \in F \atop y_1 < \|m_1\| \leq x_1} \psi(m_1) = \sum_{m_1 \in F \atop \|m_1\| \leq x_1} \psi(m_1) - \sum_{m_1 \in F \atop \|m_1\| \leq y_1} \psi(m_1) \]

which is clearly true.

• Let \( h \geq 1 \). Assume that the lemma is true for \( h - 1 \). We will prove that it remains true also for \( h \):

First we remark that

\[ \sum_{m_1, \ldots, m_h \in F \atop \|m_h\| \leq x_h \quad \forall j = 1, \ldots, h} \psi(m_1, \ldots, m_h) = \sum_{m_h \in F \atop \|m_h\| \leq x_h} \sum_{m_1, \ldots, m_{h-1} \in F \atop \|m_j\| \leq x_j \quad \forall j = 1, \ldots, h-1} \psi(m_1, \ldots, m_h) \]

\[ - \sum_{m_h \in F \atop \|m_h\| \leq y_h} \sum_{m_1, \ldots, m_{h-1} \in F \atop \|m_j\| \leq x_j \quad \forall j = 1, \ldots, h-1} \psi(m_1, \ldots, m_h). \]
The induction hypothesis implies then that

\[
\sum_{y_j < \| m_j \| \leq x_j \forall j = 1, \ldots, h} \psi(m_1, \ldots, m_h) = \sum_{m_k \in F} (-1)^{\# J} \sum_{\| m_j \| \leq y_j \forall j \in J \text{ and } \| m_j \| \leq x_j \forall j \notin J} \psi(m_1, \ldots, m_h)
\]

\[
- \sum_{m_k \in F} (-1)^{\# J} \sum_{\| m_j \| \leq y_j \forall j \in J \text{ and } \| m_j \| \leq x_j \forall j \notin J} \psi(m_1, \ldots, m_h)
\]

\[
= \sum_{I \subset \{1, \ldots, h\}} (-1)^{\# I} \sum_{m_k \in F} \psi(m_1, \ldots, m_h)
\]

\[
- \sum_{I \subset \{1, \ldots, h\}} (-1)^{\# I-1} \sum_{m_k \in F} \psi(m_1, \ldots, m_h)
\]

\[
= \sum_{I \subset \{1, \ldots, h\}} (-1)^{\# I} \sum_{m_k \in F} \psi(m_1, \ldots, m_h).
\]

This ends the proof of Lemma 6. \(\diamondsuit\)

5 Proofs of Theorems 5, 6, 7, 8 and 9

The proof of each theorem has both an analytical component, which gives a lower bound for the average of a particular metric invariant via Theorems 3, 4, and a purely diophantine geometric component that gives an upper bound for the number of integral points uniformly over an increasing family of intersections of discs and hypersurfaces. The arguments for the upper bound are identical to those given in §3 and §5 of [EL3]. As a result, we briefly sketch the particular argument needed in each case. On the other hand, we explain in detail how each of the lower bounds follows from Theorems 3, 4.

5.1 Proof of Theorem 5

Setting \(m = (m_1, m_2)\) and \(R_1(m) = \| m_1 - m_2 \|^2\), the Dirichlet series for the Distance problem is defined as follows (setting \(s = (s_1, s_2)\)) (see (3.1) in [EL3]) :

\[
Z(R_1, s) := \sum_{m=(m_1, m_2) \in (F)^2} \frac{R_1(m)}{\| m_1 \|^s_1 \| m_2 \|^s_2}.
\]

We denote

\( \mathcal{D} i = \text{the distinct distance set} := \{\| m_1 - m_2 \| : m_1, m_2 \in F\} = \{\rho_j\}_j, \)

\( \mathcal{D} i(x) := \{\rho \in \mathcal{D} i : \exists (m_1, m_2) \in F^r(x) \times F^r(x) \text{ such that } \rho = \| m_1 - m_2 \|\}, \)
so that \( \text{Dist}_F(x) = \#D_i(x) \).

The averages that interest us are:

\[
A_1(x) := A(R_1, x, x) = \sum_{m_1, m_2 \in F(x)} R_1(m_1, m_2).
\]

Since

\[
R_1(x_1, x_2) := \|x_1 - x_2\|^2 = \sum_{i=1}^n x_1^{2\varepsilon_i} + \sum_{i=1}^n x_2^{2\varepsilon_i} - 2 \sum_i x_1^\varepsilon_i x_2^\varepsilon_i = \|x_1\|^2 + \|x_2\|^2 - 2(x_1, x_2),
\]

by defining \( v_1 = (2, 0) \) and \( v_2 = (0, 2) \), and by using the notations (15) and (16), we have:

1. \( V(R_1) = \{v_1, v_2\} \);
2. \( \text{Supp}(R_1, v_1) = \{(2\varepsilon_1, 0), \ldots, (2\varepsilon_n, 0)\} \) and \( \text{Supp}(R_1, v_2) = \{(0, 2\varepsilon_1), \ldots, (0, 2\varepsilon_n)\} \);
3. \( R_{1, v_1}(x) = \sum_{i=1}^n x^{2\varepsilon_i} \geq 0 \) and \( R_{1, v_2}(x) = \sum_{i=1}^n x^{2\varepsilon_i} \geq 0 \).

It is now simple to verify the sign of the iterated residue of each \( Z(R_{1, v_1}) \) at \( e F_2 + v_i \). For example:

\[
\text{Res}_{s = e F_2 + v_1} Z(R_{1, v_1}; s) = \text{Res}_{s = e F} \zeta_F(s) \sum_{i=1}^n \text{Res}_{s = e F + 2} \zeta_F(s; 2\varepsilon_i)
\]

\[
= \text{Res}_{s = e F} \zeta_F(s) \text{Res}_{s = e F + 2} \left( \sum_{i=1}^n \zeta_F(s; 2\varepsilon_i) \right)
\]

\[
= \text{Res}_{s = e F} \zeta_F(s) \text{Res}_{s = e F + 2} \zeta_F(s - 2) = (\text{Res}_{s = e F} \zeta_F(s))^2 > 0
\]

A similar computation also shows \( \text{Res}_{s = e F_2 + v_2} Z(R_{1, v_2}; s) > 0 \).

It then follows from Theorem 3 that for any \( \delta > 0 \):

\[
A_1(x) \gg_{\delta} x^{2(e_x+1) - \delta} \quad \text{as } x \to \infty.
\]

(71)

Finding an upper bound for \( A_1(x) \) is not difficult. For any \( \rho \in D_i \) set

\[
N_{\rho}(k) = \# \{(m_1, m_2) \in (F'_i)^2 : \|m_1 - m_2\| = \rho\}; \quad M_1(x) = \text{Dist}_F(x).
\]

By Cauchy-Schwartz:

\[
A_1(x) = \sum_{\rho \in D_i} N_{\rho}(x) \cdot \rho^2 \leq \|\rho^2, \ldots, \rho^2_{M_1(x)}\| \cdot \|(N_{\rho_1}(x), \ldots, N_{\rho_{M_1(x)}}(x))\|
\]

(72)

Since \( F \subset \mathbb{Z}^n \) and each \( \rho_j \ll x \), we conclude that \textit{uniformly in} \( x \in F_i(x) \) we have (see [G])

\[
\# \{m_1 \in F'_i : \|m_1 - \xi\|^2 = \rho_j^2\} \leq \# \{m_1 \in \mathbb{Z}^n : \|m_1 - \xi\|^2 = \rho_j^2\} \ll_{\kappa} \rho_j^{n-2+\kappa} \ll_{\kappa} x^{n-2+\kappa}.
\]

(73)

Definition 3 then implies that for any sufficiently small \( \eta > 0 \), we have \textit{uniformly in} \( j \):

\[
N_{\rho_j}(x) = \sum_{\xi \in F'_i} \# \{m_1 \in F'_i : \|m_1 - \xi\|^2 = \rho_j^2\} \ll_{\eta} \# F'_i \cdot x^{n-2+\eta} = O_{\eta}(x^{e_x+n-2+\eta}).
\]

(74)

Combining (71), (72), (74) gives the following for any sufficiently small \( \delta, \eta > 0 \):

\[
x^{2(e_x+1) - \delta} \ll_{\delta} A_1(x) \ll_{\eta} x^{e_x+n+\eta} \cdot M_1(x) \quad \text{when } x \to +\infty.
\]

(75)

Thus, \( e_F > n-2 \) implies \( M_1(x) \gg_{\varepsilon} x^{e_F-n+2-\varepsilon} \gg_{\varepsilon} [\# F(x)]^{1-\frac{n-2}{e_F}-\varepsilon} \).

\[\square\]
5.2 Proof of Theorem 6

The averages of interest to us are \( A_2(x) := \sum_{m_1, m_2 \in F(x)} \theta(m_1, m_2)^2 = \sum_{m_1, m_2 \in F(x)} \langle \frac{m_1}{\|m_1\|}, \frac{m_2}{\|m_2\|} \rangle^2 \).

Using the notation (20) it is clear that \( A_2(x) = A_{(2,2)}(R_2, x, x) \), where

\[
R_2(x_1, x_2) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_1^{e_i} \cdot x_2^{e_j},
\]

for which, it is clear that the vertex set \( V \) of the corresponding Dirichlet series converges absolutely in the domain \( \{ \sigma_j > e_\mathcal{F} + 2 \ | \forall j \} \). Moreover, it’s easy to see that

\[
Res_{s=\epsilon \mathcal{F}+2} Z(R_2, s) = \sum_{i=1}^{n} \sum_{j=1}^{n} (Res_{s=\epsilon \mathcal{F}+2} \zeta_{\mathcal{F}}(s, e_i + e_j))^2 \geq \sum_{i=1}^{n} (Res_{s=\epsilon \mathcal{F}+2} \zeta_{\mathcal{F}}(s, 2e_i))^2
\]

\[
\geq \frac{1}{n} (\sum_{i=1}^{n} Res_{s=\epsilon \mathcal{F}+2} \zeta_{\mathcal{F}}(s, 2e_i))^2 = \frac{1}{n} (Res_{s=\epsilon \mathcal{F}+2} \zeta_{\mathcal{F}}(s - 2))^2 = \frac{1}{n} (Res_{s=\epsilon \mathcal{F}} \zeta_{\mathcal{F}}(s))^2 > 0.
\]

We can therefore apply Theorem 4 Part 2 with \( \mu = (2, 2) \). This implies that

\[
\forall \delta > 0, \quad A_2(x) \gg x^{2(e_\mathcal{F}+2)-4-\delta} = x^{2e_\mathcal{F}-\delta} \quad \text{as} \quad x \to \infty. \quad (76)
\]

The next step uses in an essential way a result of Heath-Brown [HB] that gives a uniform upper bound for lattice points on families of nonsingular quadrics. This proves (see pg. 1190, lines 8-10 [EL3]) the (geometric in nature) upper bound:

\[
A_2(x) \ll_{\eta} x^{e_\mathcal{F}+n-2+\eta} \text{ Ang}_{\mathcal{F}}(x) \quad \text{as} \quad x \to \infty. \quad (77)
\]

Combining (76), (77) now implies \( \text{ Ang}_{\mathcal{F}}(x) \gg_{\varepsilon} \#(\mathcal{F}(x))^{1-\frac{2-\eta}{2e_\mathcal{F}}-\varepsilon} \).

\[\square\]

5.3 Proofs of Theorems 7 and 8

The proofs of the two theorems are very similar, so we give details for the proof of Theorem 7. In Remark 3 we give a brief sketch of the changes needed to prove Theorem 8.

Since we insist upon working with distinct vectors, we first specify subsets of \( \mathcal{F}^{k+1} \) to define a rooted \( k \)-configuration zeta function. Set

\[
\mathcal{F}_{k+1} := (\mathcal{F}')^{k+1}, \quad I_k = \{ i : 1 \leq u \leq k \},
\]

where \( A(f_2, x) \) denoted \( A_2(x) \).
Applying Cauchy-Schwartz gives:

\[ \Delta_i(m) = ||m_a - m_b||^2. \]

Define also

\[ \mathcal{F}^* = \mathcal{F}_{k+1} \setminus \left( \bigcup_{i \in I_k} \mathcal{F}_i \right); \quad R_3(m) = \sum_{i \in I_k} \Delta_i(m). \]

We then define (a priori formally) the rooted configuration zeta function:

\[ Z(R_3, s) := \sum_{m \in \mathcal{F}^*} \frac{R_3(m)}{\prod_{\ell=1}^{k+1} ||m_\ell||^{s_\ell}} \quad s = (s_1, \ldots, s_{k+1}). \]  

It follows that \( Z(R_3, s) \) converges absolutely in \( \mathcal{D}_F := \{ s \in C^{k+1} : \sigma_\ell > e_\ell + 2 \ \forall \ell \}. \)

An elementary exercise connects \( A_3(x) \) to the number \( \text{Root}_{k,F}(x) \) of rooted configurations.

For any \( x \), define

\[ C^*(x) := \{ t = (t_1, \ldots, t_k) : t_i = ||m_i - m_{i+1}|| \leq k \}
\]

\[ \text{for some} \ (m_1, \ldots, m_{k+1}) \in \mathcal{F}^*(x) \}; \]

\[ M_3(x) := \# C^*(x) = \text{Root}_{k,F}(x); \]

\[ \forall t \in C^*(x) \quad N^*_t(x) := \# \left( \mathcal{F}^*(x) \cap \{ (m_1, \ldots, m_{k+1}) \in \mathbb{R}^{(k+1)n} : ||m_j - m_{j+1}|| = t_j \ \forall j \} \right). \]

Let \( t_1, \ldots, t_{M_3(x)} \) denote the distinct elements of \( C^*(x) \). Writing the components of a given \( t_j = (t_{1,j}, \ldots, t_{k,j}) \in C^*(x) \), we define \( |t_j|^2 := \sum_{u=1}^{k} t_{u,j}^2. \)

Applying Cauchy-Schwartz gives:

\[ A_3(x) = \sum_{t \in C^*(x)} |t|^2 \cdot N^*_t(x) \leq ||(t_1, \ldots, t_{M_3(x)}(x))|| \cdot ||(N^*_1(x), \ldots, N^*_x(x))||. \]

A non trivial lower bound for \( A_3(x) \) follows from Theorem 3 and inclusion-exclusion.

**Lemma 7.** \( \forall \delta > 0 \quad A_3(x) \gg_\delta x^{e_F (k+1) + 2 - \delta} \quad \text{as} \ x \to \infty. \)

**Proof.** We first apply inclusion-exclusion to show that for all \( x > 1 \)

\[ A_3(x) = A(R_3, x, \ldots, x) + \sum_{\ell=1}^{k} (-1)^\ell \sum_{I \subseteq \{1, \ldots, k\}, |I| = \ell} A_I(x) \]  

where

\[ A(R_3, x, \ldots, x) := \sum_{m \in \mathcal{F}_{k+1}(x)} R_3(m) \]

and for any \( I \neq \emptyset \)

\[ A_I(x) := \sum_{m \in \mathcal{F}_{k+1}(x) \atop m_j = m_{j+1} \ \forall j \in I} R_3(m). \]  

\[ \text{we used the notation} \ A(f_3, x) \text{ for} \ A_3(x) \text{in [EL3]} \]
The changes needed to prove Theorem 8 are as follows. First, change

$$R_3(x_1, \ldots, x_{k+1}) = \sum_{j=1}^{k} \|x_j - x_{k+1}\|^2 = \sum_{j=1}^{k} \sum_{i=1}^{n} x_{j,i}^2 + \sum_{j=1}^{n} x_{j+1,i}^2 - \sum_{j=1}^{k} \sum_{i=1}^{n} x_{j,i} x_{k+1,i}.$$  

Denoting the standard unit basis of $\mathbb{R}^n$ resp. $\mathbb{R}^{k+1}$ by $\{e_1, \ldots, e_n\}$ resp. $\{e_1, \ldots, e_{k+1}\}$, defining $v_j = 2e_j$ for each $1 \leq j \leq k+1$, and using the notations (15), (16), it is easy to see that the vertex set $\mathcal{V}(R_3) = \{v_j\}^{k+1}_{j=1}$. Moreover, defining $\beta_j = 1$ if $j \leq k$, and $\beta_{k+1} = k$, we see that for each $j$:

$$R_{3,v_j}(x_1, \ldots, x_{k+1}) = \beta_j \sum_{i=1}^{n} x_{j,i}^2,$$

$$\text{Res}_{s = e_1}^{1_{k+1} + v_j} Z(R_{3,v_j}, s) = \beta_j \left( \sum_{i=1}^{n} \text{Res}_{s = e_1}^{1_{k+1} + 2e_i} \zeta_{R_3}(s, 2e_i) \right) \left( \text{Res}_{s = e_1}^{1_{k+1}} \zeta_{R_3}(s) \right)^{k-1} = \beta_j \left( \text{Res}_{s = e_1}^{1_{k+1}} \zeta_{R_3}(s) \right)^k > 0.$$  

As a result, Theorem 3 implies

$$A(R_3, x, \ldots, x) \gg \delta x^{e_{k+1} + 2 - \delta} \quad \text{as} \quad x \to \infty. \quad (83)$$

In addition, for any $I \subset \{1, \ldots, k\}$ such that $|I| = \ell \geq 1$ and any $\varepsilon > 0$, we have uniformly in $x > 1$:

$$A_I(x) \ll x^2 \sum_{m_1, \ldots, m_{k+1} \in F(x)} \prod_{m_j = m_{k+1} \forall j \in I} 1 = x^2 \prod_{m_i \in F(x)} \left( \sum_{1 \leq m_i \leq x^{e_{k+1}}} 1 \right) = x^2 \cdot (|F(x)|)^{k+1-\ell} \ll x^{2+(k+1-\ell)(e_\varepsilon + \varepsilon)}. \quad (84)$$

Combining (82)-(84) finishes the proof of Lemma 7.

The proof of an upper bound for $A_3(x)$ is argued as in [EL3] (see pgs. 1192-93). We sketch the argument for the reader’s convenience. First note that

$$\|(t_1, \ldots, t_{|M_3(x)|})\| \ll M_3(x)^{1/2} \cdot \sup |t_j|^2 \ll M_3(x)^{1/2} \cdot x^2 \quad (85)$$

Second, use (73) to bound each $N_{t_j}(x)$ uniformly in $t_j$ as follows:

$$N_{t_j}(x) \leq \sum_{m_1, \ldots, m_{k+1} \in F^*(x)} \prod_{|m_i - m_{k+1}| = t_{i,j}} 1 \leq \prod_{|m_i - m_{k+1}| = t_{i,j}} \sum_{m_i \in F^*(x)} 1 \ll x^{e_{k} + k(n-2) + \eta}.$$  

Lemma 7 and (81) then imply that $x^{e_{k} + k+2 - \delta} \ll x^{e_{k} + k(n-2) + 2 + \eta} \cdot M_3(x)$. Thus, $M_3(x) \gg \varepsilon \cdot |F(x)|^{k+1-\eta} \cdot x^{2+2\varepsilon}$ as $x \to \infty$. \hfill $\Box$

**Remark 3.** The changes needed to prove Theorem 8 are as follows. First, change $I_k$ to $J_k = \{i, i+1 : 1 \leq i \leq k\}$. Then set $F_\ast = F_{k+1} \setminus \cup_{l \in J_k} F_l$, $F_\ast(x) = F_\ast \cap F_{k+1}(x)$, and

$$R_4(m) := \sum_{i \in J_k} \Delta_i(m); \quad Z(R_4, s) := \sum_{m \in F_\ast} \frac{R_4(m)}{\prod_{i=1}^{k+1} \|m_i\|^s}; \quad A_4(x) := \sum_{m \in F_\ast(x)} R_4(m).$$
Note that $Z(R_4, s)$ converges absolutely in $\mathcal{D}_F$. As in (80)ff., define:

$$
\mathcal{C}_s(x) := \{t = (t_1, \ldots, t_k) : t_i = \|m_i - m_{i+1}\| \forall i \leq k \text{ for some } m \in \mathcal{F}_s(x)\},
$$

$$
M_4(x) := \#\mathcal{C}_s(x) = \#\text{Chain}_{k,\mathcal{F}}(x),
$$

$$
\forall t \in \mathcal{C}_s(x) \quad N_{s,t}(x) := \#\left(\mathcal{F}_s(x) \cap \{(m_1, \ldots, m_{k+1}) \in \mathbb{R}^{(k+1)n} : \|m_j - m_{j+1}\| = t_j \forall j\right) ,
$$

$$
\mathcal{A}_4(x) := \sum_{t \in \mathcal{C}_s(x)} |t|^2 \cdot N_{s,t}(x) \leq \|(t_1^2, \ldots, |t_{M_4(x)}|^2)\| \cdot \|(N_{s,t_1}(x), \ldots, N_{s,t_{M_4(x)}}(x))\|.
$$

An argument similar to the proof of Lemma 7 shows

**Lemma 8.** For each $\varepsilon > 0 \quad \mathcal{A}_4(x) \gg_{\varepsilon} x^{eF(k+1)+2-\varepsilon} \quad \text{as } x \to \infty.$

**Proof of Lemma 8:** Set $\gamma_j = 2 \forall j = 2, \ldots, k$ and $\gamma_1 = \gamma_{k+1} = 1$. Then we have

$$
R_4(X_1, \ldots, X_{k+1}) = \sum_{j=1}^{k+1} \|X_j - X_{j+1}\|^2 = \sum_{j=1}^{k+1} \gamma_j \sum_{i=1}^{n} X_{j,i}^2 - 2 \sum_{j=1}^{k} \sum_{i=1}^{n} X_{j,i}X_{j+1,i}.
$$

Denoting the canonical bases of $\mathbb{R}^n$ resp. $\mathbb{R}^{k+1}$ by $\{e_1, \ldots, e_n\}$ resp. $\{e_1, \ldots, e_{k+1}\}$, defining $v_j = 2e_j$ for each $1 \leq j \leq k+1$, and using the notations from Theorem 3, it is easy to see that the vertex set $V(R_4) = \{v_j\}_{j=1}^{k+1}$ and $R_4(v_j, X_1, \ldots, X_{k+1}) = \gamma_j \sum_{i=1}^{n} X_{j,i}^2$ for all $j = 1, \ldots, k+1$. The remainder of the proof is identical to the proof of Lemma 7.

**Proof of Theorem 8:** In view of Lemma 8, we only need an upper bound for $\mathcal{A}_4(x)$. First note that (85) extends immediately: $\|(t_1^2, \ldots, |t_{M_4(x)}|^2)\| \ll x^2 \cdot M_4(x)^{1/2}.$

Iterating $k$ times the estimate (73), we bound each $N_{s,t_j}(x)$ uniformly in $t_j$ as follows:

$$
N_{s,t_j}(x) \leq \sum_{m_1, \ldots, m_{k+1} \in \mathcal{F}'(x), \|m_i - m_{i+1}\| = t_j} 1 \leq \sum_{m_1, \ldots, m_{k} \in \mathcal{F}'(x), \|m_i - m_{i+1}\| = t_j} \left( \sum_{m_{k+1} \in \mathcal{F}'(x), \|m_{k+1} - m_k\| = t_{k+1}} 1 \right) 
$$

$$
\ll_{\varepsilon} t_j^{n-2+\varepsilon} \sum_{m_1, \ldots, m_{k} \in \mathcal{F}'(x), \|m_i - m_{i+1}\| = t_j} 1 \ll_{\varepsilon} (t_{k+1}t_{k-1,1})^{n-2+\varepsilon} \sum_{m_{k+1} \in \mathcal{F}'(x), \|m_{k+1} - m_k\| = t_{k+1}} 1
$$

$$
\ll_{\varepsilon} \ldots \ll_{\varepsilon} (t_{k,j} \ldots t_{1,j})^{n-2+\varepsilon} \sum_{m_{1} \in \mathcal{F}'(x)} 1 \ll_{\varepsilon} (t_{k,j} \ldots t_{1,j})^{n-2+\varepsilon} e^{eF+\varepsilon} \ll_{\varepsilon} x^{eF+k(n-2)+\varepsilon}.
$$

The remaining details are identical to those that finished the proof of Theorem 7.

**5.4 Proof of Theorem 9**

We first define (formally) the determinant zeta function associated to $\mathcal{F}$ as follows:

$$
\zeta_{det}(\mathcal{F}; s) := \sum_{m_1, \ldots, m_n \in \mathcal{F}} \frac{det^2(m_1, \ldots, m_n)}{\|m_1\|^{s_1} \cdots \|m_n\|^{s_n}}, \quad s = (s_1, \ldots, s_n). \quad (87)
$$

Setting $D_F = e_F + 2$, applying Hadamard’s inequality and formula (4.4) of ([EL3], p. 1194), we note that Theorem 2 implies the following.
Proposition 2. The determinant zeta function $\zeta_{det}(\mathcal{F},s)$ of $\mathcal{F}$ converges absolutely in the domain $\cap_{i=1}^{n} \{ \sigma_i > D_F \}$ and has a meromorphic extension (also denoted $\zeta_{det}(\mathcal{F},s)$) with moderate growth to a domain $\cap_{i=1}^{n} \{ \sigma_i > D_F - \eta \}$ for some $\eta > 0$.

Definition 6. The set $\mathcal{F}$ is thick if the point $(D_F, \ldots, D_F)$ is a pole of $\zeta_{det}(\mathcal{F},s)$ and the iterated residue $\text{Res}_{s_1=D_F} \cdots \text{Res}_{s_n=D_F} \left( \zeta_{det}(\mathcal{F},s) \right) \neq 0$.

Define $R_5(X_1, \ldots, X_{n+1}) := \det^2(X_1 - X_{n+1}, \ldots, X_n - X_{n+1})$, $\mathbf{s} = (s, s_{n+1})$, and the "simplex" 4 zeta function

$$Z(R_5; \mathbf{s}) = \sum_{m_1, \ldots, m_{n+1} \in \mathcal{F}} \frac{R_5(m_1, \ldots, m_{n+1})}{|m_1|^s \cdots |m_{n+1}|^{s_{n+1}}}. \quad (88)$$

In addition, define $A_5(x) := A(R_5, x, \ldots, x) = \sum_{m_1, \ldots, m_{n+1} \in \mathcal{F}(x)} R_5(m_1, \ldots, m_{n+1})$, and the points $v_u = 2(1_{n+1} - e_u) \in \mathbb{R}^{n+1} := (v_1, u_1, \ldots, v_{n+1}, u)$ for each $u = 1, \ldots, n+1$.

Applying formulae (4.5), (4.6) in [EL3] shows the following (see (15)).

$$\mathcal{V}(R_5) = \{v_1, \ldots, v_{n+1}\}; \quad R_{5,v_{n+1}}(X) = \det^2(X_1, \ldots, X_n); \quad R_{5,v_u}(X) = \det^2(X_1, \ldots, X_{n-1}, X_{n+1}, X_{u+1}, \ldots, X_n) \text{ if } u \leq n.$$ 

Moreover, defining $\mathbf{s}_u = (s_1, \ldots, s_{u-1}, s_{u+1}, \ldots, s_{n+1})$ for each $u \leq n + 1$, the hypothesis that $\mathcal{F}$ is thick implies that for any $u$:

$$\text{Res}_{s=e_F} 1_{n+1} + v_u Z(R_5,v_u; s) = \text{Res}_{s=D_F} 1_{n+1} - 2v_u Z(R_5,v_u; s)$$

$$= \text{Res}_{s=e_F} \zeta_{\mathcal{F}}(s_u) \cdot \left( \text{Res}_{s_1=D_F} \cdots \text{Res}_{s_{u-1}=D_F} \text{Res}_{s_{u+1}=D_F} \cdots \text{Res}_{s_{n+1}=D_F} (\zeta_{det}(\mathcal{F}; \mathbf{s}_u)) \right)$$

$$= \text{Res}_{s=e_F} \zeta_{\mathcal{F}}(s) \left( \text{Res}_{s_1=D_F} \cdots \text{Res}_{s_{n+1}=D_F} (\zeta_{det}(\mathcal{F}; s)) \right) \neq 0.$$ 

Applying Theorem 3 it follows that

$$A_5(x) \gg_{\delta} x^{(n+1)e_F + 2n - \delta} \text{ as } x \to \infty. \quad (89)$$

In [EL3] (§5, formula 5.15), we proved the following upper bound

$$\forall \varepsilon > 0, \quad A_5(x) \ll_{\varepsilon} x^{3n-1+ne_F+\varepsilon} \text{Vol}_{n,\mathcal{F}}(x). \quad (90)$$

Combining (90) with (89) then implies $\text{Vol}_{n,\mathcal{F}}(x) \gg_{\varepsilon} x^{e_F-(n-1)-\varepsilon} \gg_{\varepsilon} |\mathcal{F}(x)|^{1-\frac{n-1}{e_F} - \varepsilon}. \quad \square$

6 Proofs of Theorems 10, 11 and 12

Proof of Theorem 10: Set $R^{2\alpha}(X_1, \ldots, X_k) = \prod_{j=1}^{k} X_j^{2\alpha_j}$ and $\mathbf{v} = (2|\alpha_1|, \ldots, 2|\alpha_k|)$.

Using notations from Theorem 3, it is easy to see that

$$\mathcal{V}(R^{2\alpha}) = \{\mathbf{v}\}; \quad R^{2\alpha}_v = R^{2\alpha} \geq 0; \quad \text{Res}_{s=e_F} 1_k + v \cdot Z(R^{2\alpha}_v; s) = \prod_{j=1}^{k} \text{Res}_{s=e_F + 2|\alpha_j|} \zeta_{\mathcal{F}}(s, 2\alpha_j) \neq 0.$$

\footnote{4Parallelopetope" would be more precise but is too polysyllabic.}
\footnote{We used the notation $A(f_s, x)$ for $A_5(x)$ in [EL3].}
By Theorem 3 it follows that for any $\delta > 0$,

$$A(x) := \sum_{m_1, \ldots, m_k \in \mathcal{F}(x)} R^{2\alpha}(m_1, \ldots, m_k) \gg_{\delta} x^{|\alpha| + \delta} \text{ as } x \to \infty. \quad \text{(91)}$$

Defining $n_\alpha(x) = \lvert \mathcal{F}(x)^\alpha \rvert$, $\mathcal{F}(x)^\alpha = \{0 < \rho_2 < \cdots < \rho_{n_\alpha(x)}\}$ and

$$N_j := \#\{ (m_1, \ldots, m_k) \in \mathcal{F}(x)^k : \prod_{\ell=1}^k m_\ell^{\alpha_i} = \rho_j \} \quad (1 \leq j \leq n_\alpha(x)),$$

it is clear that: $A(x) = \sum_{j=1}^{n_\alpha(x)} \rho_j^2 N_j$.

Since each $m_{\ell,i} \in [-x,x]$, we have for any $j = 1, \ldots, n_\alpha(x)$ such that $\rho_j \neq 0$:

$$N_j \leq \#\{ (m_{1,1}, \ldots, m_{1,n}, \ldots, m_{k,1}, \ldots, m_{k,n}) \in \mathbb{Z}^{kn} \cap [-x, x]^{kn} : \prod_{\ell=1}^k m_{\ell,i}^{\alpha_{\ell,i}} = \rho_j \}$$

$$\leq \#\{ (m_{1,1}, \ldots, m_{1,n}, \ldots, m_{k,1}, \ldots, m_{k,n}) \in \mathbb{Z}^{kn} : m_{\ell,i} \in [-x, x] \text{ if } \alpha_{\ell,i} = 0 \text{ and } m_{\ell,i}|\rho_j \text{ if } \alpha_{\ell,i} \neq 0 \}$$

$$\ll x^{u(\alpha)} (d(\rho_j))^{kn-u(\alpha)},$$

where $d(\rho)$ denotes the number of divisors of $\rho$.

In addition, the standard bound $d(\rho) \ll_{\varepsilon} |\rho|^\varepsilon$ implies the following estimate for each $\varepsilon > 0$ that is uniform in $x, j$:

$$N_j \ll_{\varepsilon} x^{u(\alpha)} |\rho_j|^\varepsilon. \quad \text{(92)}$$

Since $|\rho_j| \leq x^{\lvert \alpha \rvert}$, we conclude that for any $\varepsilon > 0$

$$A(x) \ll_{\varepsilon} n_\alpha(x) \cdot x^{2\lvert \alpha \rvert + u(\alpha) + \varepsilon}. \quad \text{(93)}$$

Combining (91) and (93) it follows that for any $\varepsilon, \delta > 0$, we have for all sufficiently large $x$:

$$x^{k_{e_F} x + 2|\alpha| - \delta} \ll_{\delta} A(x) \ll_{\varepsilon} n_\alpha(x) \cdot x^{2|\alpha| + u(\alpha) + \varepsilon}.$$ 

Thus, for all $\varepsilon > 0$, $n_\alpha(x) \gg_{\varepsilon} x^{k_{e_F} x - u(\alpha) - \varepsilon} \gg_{\varepsilon} \lvert \mathcal{F}(x)^\alpha \lvert_{k_{e_F} x - u(\alpha) - \varepsilon} \gg_{\varepsilon} \lvert \mathcal{F}(x)^{k_{e_F} x - u(\alpha) + \varepsilon}$. \quad \text{(94)}$

This finishes the proof of Theorem 10. \hfill \Box$

**Proof of Theorem 11:**

Denote by $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$ the standard unit basis vectors of $\mathbb{R}^n$.

Since $\sigma > 0$ implies $\sum_{i=1}^n \zeta_F(e_F + 2 + \sigma, 2e_i) = \zeta_F(e_F + \sigma)$, by letting $\sigma \to 0+$, it follows that

$$\sum_{i=1}^n Res_{s = e_F + 2} \zeta_F(s, 2e_i) = Res_{s = e_F} \zeta_F(s) \neq 0.$$ Thus, there exists $a \in \{1, \ldots, n\}$ such that $Res_{s = e_F + 2} \zeta_F(s, 2e_a) \neq 0$.

For this index $a$, we now set $\alpha = (e_a, e_a, \ldots, e_a)$. From (94), we conclude that for any $\varepsilon > 0$, $|\lvert \pi_a \mathcal{F}(x) \rvert| = n_\alpha(x) \gg_{\varepsilon} x^{k_{e_F} x - k(n-1) - \varepsilon}$. 32
Moreover, since $\pi_a \mathcal{F}(x) \subset \mathbb{Z} \cap [-x, x]$ it is clear that $|\pi_a \mathcal{F}(x)| \ll x$. Thus, for any $\varepsilon > 0$, 
$(|\pi_a \mathcal{F}(x)|^k)^\varepsilon \gg |\pi_a \mathcal{F}(x)|^{k(e_x + 1 - n) - \varepsilon}$. This finishes the proof of Theorem 11. $\diamond$

**Proof of Theorem 12:**

Set $R(X_1, \ldots, X_k) := (P(X_1, \ldots, X_k))^2 = \sum_{\alpha, \beta \in \text{Supp}(P)} a(\alpha)a(\beta)X_1^{\alpha_1+\beta_1} \ldots X_k^{\alpha_k+\beta_k}$.

For each $j = 1, \ldots, k$, set $v_j := 2d \varepsilon_j$ and $P_j(X_j) := \sum_{\alpha, \beta \in S_j(P)} a(\alpha)a(\beta)X_j^{\alpha_j+\beta_j}$. By (15) and (16), it is easy to see that the vertex set is $\mathcal{V}(R) = \{v_1, \ldots, v_k\}$ and $R_{v_j} = P_j$ for each $j$. Thus, $R_{v_j} = \left(\sum_{\alpha \in S_j(P)} a(\alpha)X_j^{\alpha_j}\right)^2 \geq 0$, and

$Res_{\varepsilon X + v_j} Z(R_{v_j}, s) = (Res_{\varepsilon X} \zeta_{\mathcal{F}}(s))^k \left(\sum_{\alpha, \beta \in S_j(P)} a(\alpha)a(\beta)Res_{\varepsilon X + 2d \varepsilon_j} \zeta_{\mathcal{F}}(s; \alpha_j + \beta_j)\right) \neq 0$.

Theorem 3 now implies that for any $\delta > 0$,

$A_R(x) := \sum_{m_1, \ldots, m_k \in \mathcal{F}(x)} R(m_1, \ldots, m_k) \gg_{\delta} x^{k_\varepsilon x + degR - \delta} = x^{k_\varepsilon x + 2d - \delta}$ as $x \to \infty$. (95)

Defining $n_P(x) := \{\rho_1 < \rho_2 < \cdots < \rho_{n_P(x)}\} := P\mathcal{F}(x)$, and $N_j(x) := \# \{(m_1, \ldots, m_k) \in \mathcal{F}(x)^k : P(m_1, \ldots, m_k) = \rho_j\}$, we have $A_R(x) = \sum_{j=1}^{n_P(x)} \rho_j^2 N_j(x)$. Moreover, since each $m_j \in \mathbb{Z}^n \cap [-x, +x]^n$, it follows that

$N_j(x) \leq \# \{(m_1, \ldots, m_k) \in \mathbb{Z}^{kn} \cap [-x, +x]^{kn} : P(m_1, \ldots, m_k) = \rho_j\}$.

As in the proof of [EL4, Lemma 1], a result of Browning-Heath-Brown [BHB] now implies:

$\forall \varepsilon > 0 \quad N_j(x) \ll_{\varepsilon} x^{kn-1+\varepsilon}$ uniformly in $x$ and $j$. (96)

In addition, since $|\rho_j| \ll x^d$, it is clear that for any $\varepsilon > 0$

$A_R(x) = \sum_{j=1}^{n_P(x)} \rho_j^2 N_j(x) \ll_{\varepsilon} n_P(x) \cdot x^{2d + kn - 1 + \varepsilon}$. (97)

Combining (95) and (97), we conclude

$\forall \varepsilon > 0 \quad n_P(x) \gg_{\varepsilon} x^{k_\varepsilon (e_x - n) + 1 - \varepsilon} \gg_{\varepsilon} |\mathcal{F}(x)|^{k_{e_x - n} + 1 - \varepsilon} x^{\varepsilon |\mathcal{F}(x)|^{k_{e_x - n} + 1 - \varepsilon}}\gg_{\varepsilon}$, which finishes the proof of Theorem 12.

$\square$
References


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