THÈSE

Pour l'obtention du grade de

Docteur de l'Université de Poitiers

(Faculté des Sciences Fondamentales et Appliquées)

(Diplôme National - Arrêté du 25 Avril 2002)

Ecole Doctorale : Sciences pour l'Ingénieur

Secteur de Recherche : Mathématiques et leurs intéractions

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(Co)homologie de Poisson et singularités isolées en petites dimensions, avec une application en théorie des déformations

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Remerciements

Certaines personnes sont capables de vous transmettre une partie de leurs connaissances avec tant de patience, d'enthousiasme et de générosité qu'elles vous transforment et transforment votre vie, dans le bon sens. Ce qu'elles vous offrent alors n'a pas de prix, à mon sens. Je ne peux que remercier chaleureusement les personnes ayant tenu ce rôle à un moment ou un autre de ma vie, sans jamais espérer, malheureusement, réussir à leur rendre une partie de ce qu'elles m'ont donné.

Pol Vanhaecke est de celles-là. Je suis contente de pouvoir ici le remercier d'avoir été un formidable directeur de thèse, aussi bien sur le plan scientifique que sur le plan humain. Seules quelques personnes savent à quel point j'ai été enthousiaste et fière de travailler avec lui. Je lui serai toujours reconnaissante de m'avoir initiée à de si "jolies" mathématiques et, quand je serai "grande" et que je serai un peu moins son étudiante en thèse, je serai flattée si je peux compter parmi ses amis.

Jacques Alev et Ping Xu ont bien voulu effectuer le lourd et important travail qui revient aux rapporteurs. Je tiens à les en remercier, de même que pour l'intérêt qu'ils portent ainsi à mon travail.

Camille Laurent-Gengoux, Claude Roger, Fred Van Oystaeyen et Nguyen Tien Zung ont tous accepté d'être membres de mon jury. Je les remercie chaleureusement et je suis honorée de soutenir ma thèse devant eux.

Ces quatre années de doctorat ont été l'occasion de nombreuses rencontres qui ont enrichi le travail que je présente aujourd'hui ou m'ouvrent de belles perspectives pour la suite. Parmi les personnes qui m'ont aidée, Jacques Alev, Pantelis Damianou, Rui Fernandes, Benoit Fresse, Viktor Ginzburg, Jiang-Hua Lu, Michael Penkava, Claude Quitté (qui, grâce à sa connaissance des suites régulières, a considérablement apporté à mon travail), Camille Laurent-Gengoux (qui m'a appris, entre autres, la classe modulaire et avec qui j'espère encore apprendre beaucoup d'-oids), Thierry Lambre, Olivier Mathieu (dont le chapitre 5 est une réponse à l'une de ses questions), Philippe Monnier, Alan Weinstein, sans oublier Eva Miranda avec qui j'ai eu quelques discussions passionnantes. Je remercie également tous les doctorants que j'ai rencontrés, soit à Poitiers, soit au cours de conférences ou de réunions CIES et avec qui j'ai beaucoup appris.

Mes années en tant que thésarde ont été les plus belles années de mes études et beaucoup de personnes ont contribué à ce fait. Tout d'abord, l'ambiance au Laboratoire de Mathématiques et Applications de Poitiers aura été très agréable et respectueuse. En particulier, Abderrazak Bouaziz, Alain Miranville, Hervé Sabourin, Patrice Tauvel (entre autres) auront considérablement contribué au plaisir que j'avais de venir au laboratoire tous les matins, je les remercie pour cela ainsi que pour m'avoir souvent aidée et conseillée. Je tiens également à remercier Jocelyne Attab, Brigitte Brault, Nathalie Marlet, Valérie Menet et Benoit

Métrot pour leur aide, bien souvent indispensable, et leur constant sourire. Bien sûr, rien n'aurait été pareil sans la rencontre de ceux que l'on appelle désormais les fameux "SVS-istes", qui ont rendu ces trois années bien agréables et avec lesquels j'ai partagé beaucoup de très bons moments : Anthony, Alice, Camille, Céline, Hélène, Ioan, Mathieu, Morgan, Paul, Rupert ... Enfin, je voudrais remercier ceux avec qui j'ai eu la chance de partager quelques mois ou quelques années d'études, un bureau... et, à chaque fois, beaucoup plus : Ariane, Gregory ainsi que Khaoula et Patience à qui je souhaite de nombreux bons moments dans la suite de leur thèse.

Mon idée n'est pas ici de citer tous mes amis, je voudrais encore simplement rajouter quelques lignes pour Gwénaëlle, Geoffroy et Ophélie, qui, généreusement, m'ont accueillie (voire recueillie!) très souvent, j'ai passé chez eux des moments très agréables de sérénité et d'amitié; Julien, mon ami; Julie, avec qui je parle, avec plaisir, de beaucoup de choses mais pas de maths! Et enfin, Nathalie qui m'a certainement apporté beaucoup plus qu'elle ne se l'imagine. Elle aussi fait partie de ces personnes dont j'ai eu la chance d'apprendre et dont je parlais plus haut. Ce qu'elle sait, en revanche, je le crois, c'est à quel point je lui ai fait confiance, tellement j'avais toujours l'impression de mettre ma vie entre ses mains, au début de chaque cours d'équitation et de la reprendre à la fin...!

Je ne suis jamais seule. Quelque soit le moment de ma vie, les difficultés à affronter ou les bonnes nouvelles à accueillir, les moments de doute ou de joie, ma famille m'accompagne. Nous formons une équipe et leur soutien sans faille et sans condition est pour moi source d'une très grande force. Je remercie ma mère, mes deux frères et ma grand-mère, simplement d'être là. Et je ne soutiendrai pas cette thèse sans avoir une pensée toute particulière pour mon grand-père et mon père.

Je m'étais bien promis d'être concise mais..

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1

Introduction

The first Poisson structures appeared in classical mechanics. In 1809, D. Poisson introduced a bracket of functions, given by:

$$\{F, G\} = \sum_{i=1}^{r} \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right), \tag{1.1}$$

for two smooth functions F, G on $\mathbf{R}^{2r}$. It permits one to write the Hamilton's equations as differential equations, where positions (q_i) and momenta (p_i) play symmetric roles. Indeed, denoting by H the total energy of the system, these equations can be written as:

$$\begin{aligned} \dot{q}_i &= \{q_i, H\}, \\ \dot{p}_i &= \{p_i, H\}, \end{aligned} \qquad 1 \leq i \leq r.$$

Poisson's crucial observation was that if F and G are constants of motion, then $\{F, G\}$ is also a constant of motion and this phenomenon was explained in 1839 by C. Jacobi, who proved that (1.1) satisfies what is now called the Jacobi identity:

$$\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0. \tag{1.2}$$

This important identity leads to the definition of Poisson manifolds, that were introduced by A. Lichnerowicz, in [40], as a generalization of symplectic manifolds.

Poisson varieties

A *Poisson manifold* (see Section 2.1 for more details) is a manifold M whose algebra of smooth functions $\mathcal{F}(M) = C^\infty(M)$ is equipped with a skew-symmetric bilinear map

$$\{\cdot\,, \cdot\} : \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M),$$

satisfying

1. The Leibniz rule (derivation property),

$$\{FG, H\} = F\{G, H\} + \{F, H\} G, \quad F, G, H \in \mathcal{F}(M); \tag{1.3}$$

2. The Jacobi identity (1.2), for all $F, G, H \in \mathcal{F}(M)$.

Said differently, a manifold M is a Poisson manifold if its algebra of (smooth) functions $\mathcal{A} = \mathcal{F}(M)$ is endowed with a Lie bracket $\{\cdot\,,\cdot\}$ that satisfies the Leibniz rule. Similarly, one defines an *affine Poisson variety*, by considering the algebra of regular functions, instead of the algebra of smooth functions.

Let us give some examples of affine Poisson varieties or Poisson manifolds. As we suggested previously, a symplectic manifold is naturally a Poisson manifold. For a symplectic manifold (M, ω), one defines indeed a Poisson bracket, with the formula

$$\{F, G\} = \omega(\chi_F, \chi_G), \quad F, G \in C^\infty(M),$$

where the vector field χ_F is defined by $\mathsf{d}F = \omega(\chi_F, \cdot)$.

Another classical example of Poisson manifold is the dual $\mathfrak{g}^*$ of a finite dimensional Lie algebra $\mathfrak{g}$. For $F \in C^\infty(\mathfrak{g}^*)$ and $\xi \in \mathfrak{g}^*$, $\mathsf{d}F(\xi)$ is an element of the bidual $\mathfrak{g}^{**} \simeq \mathfrak{g}$ and the formula

$$\{F, G\}(\xi) = \xi\Big([\mathsf{d}F(\xi), \mathsf{d}G(\xi)]\Big), \quad F, G \in C^\infty(\mathfrak{g}^*),\ \xi \in \mathfrak{g}^*,$$

defines a Poisson structure $\{\cdot\,,\cdot\}$ on $\mathfrak{g}^*$. Taking as algebra of functions on $\mathfrak{g}^*$, the algebra of polynomial functions, $\mathrm{Sym}(\mathfrak{g})$, instead of $C^\infty(\mathfrak{g}^*)$, the above formula defines a structure of affine Poisson variety on $\mathfrak{g}^*$.

Another example is given by considering a smooth function $\psi \in C^\infty(\mathbf{F}^2)$ on the manifold $\mathbf{F}^2$ ($\mathbf{F}$ is $\mathbf{R}$ or $\mathbf{C}$). Then, the bracket $\{\cdot\,,\cdot\}$, defined by:

$$\{F, G\} = \psi\left(\frac{\partial F}{\partial x}\frac{\partial G}{\partial y} - \frac{\partial F}{\partial y}\frac{\partial G}{\partial x}\right), \quad F, G \in C^\infty(\mathbf{F}^2),$$

is a Poisson structure on $\mathbf{F}^2$. By replacing $C^\infty(\mathbf{F}^2)$ by the algebra of polynomial functions $\mathbf{F}[x, y]$ on $\mathbf{F}^2$ (where $\mathbf{F}$ becomes an arbitrary field of characteristic zero), we then obtain an affine Poisson variety $(\mathbf{F}^2, \{\cdot\,,\cdot\})$.

Poisson cohomology

In [40], A. Lichnerowicz has also introduced a cohomology, associated to a Poisson structure, named *Poisson cohomology*; see also [32] for an algebraic approach. For $(M, \{\cdot\,,\cdot\})$ an affine Poisson variety and $\mathcal{A} = \mathcal{F}(M)$ its algebra of regular functions, the Poisson complex is defined as follows. The cochains are the skew-symmetric multilinear maps $\mathcal{A} \times \cdots \times \mathcal{A} \to \mathcal{A}$, satisfying the Leibniz rule in each of their arguments, as in (1.3). Such maps are called skew-symmetric multiderivations of the algebra $\mathcal{A} = \mathcal{F}(M)$ and the space of all skew-symmetric multiderivations is denoted by $\mathfrak{X}^\bullet(\mathcal{A})$. In the case of a manifold, the skew-symmetric multiderivations of $C^\infty(M)$ are in natural one-to-one correspondence with the polyvector fields on M. The Poisson coboundary operator δ is the $\mathbf{F}$-linear map, defined for $Q \in \mathfrak{X}^q(\mathcal{A})$, a skew-symmetric q-derivation of $\mathcal{A}$, and for $F_0, \ldots, F_q \in \mathcal{A}$, by

$$\delta^q(Q)(F_0,\dots,F_q) := \sum_{i=0}^{q}(-1)^i \left\{F_i, Q(F_0,\dots,\widehat{F_i},\dots,F_q)\right\} \\ + \sum_{0\le i<j\le q}(-1)^{i+j}Q\left(\{F_i,F_j\}, F_0,\dots,\widehat{F_i},\dots,\widehat{F_j},\dots,F_q\right).$$

We can also write this coboundary operator as $\delta = -\left[\pi,\cdot\right]_S$, where $\pi := \{\cdot\,,\cdot\}$ is the Poisson bracket and $[\cdot\,,\cdot]_S$ is the Schouten bracket.

The resulting Poisson complex, defined in detail in Section 2.2, can be viewed as the contravariant version of the de Rham complex. Its cohomology gives very interesting information about the Poisson structure, as for small k, the k-th Poisson cohomology space $H^k(M,\pi)$ has the following interpretation:

$$H^0(M,\pi) = \mathrm{Cas}(M,\pi) := \{F \in \mathcal{F}(M) \mid \{F,\,\cdot\,\} = 0\},$$

$$H^1(M,\pi) = \frac{\{\text{Poisson derivations}\}}{\{\text{Hamiltonian derivations}\}},$$

$$H^2(M,\pi) = \frac{\{\text{skew-symmetric biderivations compatible with } \pi\}}{\{\text{Lie derivatives of } \pi\}},$$

$$H^3(M,\pi) = \{\text{Obstructions to deformations of Poisson structures}\},$$

where the elements of $\mathrm{Cas}(M,\pi)$ (elements of the center of $\{\cdot\,,\cdot\}$) are called the *Casimirs* of $(M,\{\cdot\,,\cdot\})$, while the *Hamiltonian derivations* are the derivations of the form $\chi_H := \{\cdot\,,H\}$, for $H \in \mathcal{A}$ and the *Poisson derivations* are the derivations that leave the Poisson structure invariant.

Moreover, $H^2(M,\pi)$ is important in the study of normal forms and for the linearization of Poisson structures (see [19] and [12,13]) . Both spaces $H^2(M,\pi)$ and $H^3(M,\pi)$ play a fundamental role in deformation theory (see Chapter 6). They appear in the construction of deformations of Poisson structures and in the classification of deformations of associative structures (see [34]).

To determine the Poisson cohomology of a given affine Poisson variety (or Poisson manifold) explicitly is, in general, difficult. One of the reasons seems to be that Poisson cohomology is not a functor: a morphism $\varphi : M_1 \to M_2$ between affine Poisson varieties does not lead to a morphism between their Poisson cochains (multiderivations), nor between their corresponding Poisson cohomology groups.

In a few specific cases, Poisson cohomology has been determined. For a symplectic manifold, there exists a natural isomorphism between Poisson and de Rham cohomology (see [40] and also [39]), thus, in this case, Poisson cohomology give topological information about the underlying manifold. In [59,64], one finds some partial results about the more general case of regular Poisson manifolds, while, for Poisson-Lie groups, one can refer to [29]. The Poisson cohomology in dimension two was computed in [59] for the manifold $\mathbf{R}^2$, equipped with the Poisson structure $(x^2+y^2)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}$, in [47] for quadratic Poisson structures on $\mathbf{R}^2$, in [45], where the author has computed an explicit basis for the Poisson cohomology spaces in a germified case, while, in [54], the authors compute the dimensions of the Poisson cohomology spaces, associated to homogeneous Poisson brackets on the affine space $\mathbf{F}^2$, equipped with its algebra of regular polynomial functions.

Singular locus of Poisson brackets and singular varieties

For an affine Poisson variety $(M \subseteq \mathbf{F}^n, \{\cdot\,,\cdot\})$ and $m \in M$, one denotes by

$$\mathrm{Ham}_m(M) := \{\chi_H(m) \mid H \in \mathcal{F}(M)\}$$

the $\mathbf{F}$-vector space of all the Hamiltonian derivations, evaluated at m. One defines the *rank* $\mathrm{Rk}_m\{\cdot\,,\cdot\}$ of $\{\cdot\,,\cdot\}$ at m, with:

$$\mathrm{Rk}_m\{\cdot\,,\cdot\} := \dim \mathrm{Ham}_m(M).$$

Then, one says that the bracket $\{\cdot\,,\cdot\}$ is *regular* at the point m if there exists a (Zariski) open neighborhood V of m, on which the rank is constant, i.e., such that, for all $m' \in V$, one has $\mathrm{Rk}_{m'}(M) = \mathrm{Rk}_m(M)$. Otherwise, one says that $\{\cdot\,,\cdot\}$ is *singular* on m, or that m is a *singularity* of $\{\cdot\,,\cdot\}$.

The *singular locus* of a Poisson bracket is the set of all its singularities and it defines an affine subvariety of M (i.e. a Zariski closed subset of M). If this singular locus is empty, one says that the Poisson bracket is regular and otherwise, one says that it is singular.

In the particular case of $(\mathbf{F}^2, \{\cdot\,,\cdot\}^\psi)$, the singular locus of the bracket $\{\cdot\,,\cdot\}^\psi$ coincides with the zero locus of the polynomial ψ, while, for example, in the symplectic case, the Poisson bracket is regular.

We are now in position to state the purpose of this thesis: to study Poisson cohomology and homology in singular cases. The singularities originate in different classes of Poisson varieties:

(1) the first class consists of Poisson structures that admit a singular locus and that are defined on a smooth (regular) affine variety;

(2) the second class is the case of Poisson brackets, defined on singular varieties and regular everywhere, except on the singularities of the variety.

An important part of our work consists of considering Poisson varieties from both classes (1) and (2), in low dimension, i.e., in dimension two and three and of determining their Poisson cohomology and homology.

In dimension three

In the class (1) of Poisson varieties (where the affine variety is smooth but the Poisson structure is singular), we will, in Chapter 3, consider the affine space of dimension three $\mathbf{F}^3$ ($\mathbf{F}$ is a field of characteristic zero) and its algebra of regular functions $\mathcal{A} := \mathbf{F}[x, y, z]$, equipped with a Poisson structure that has the origin $0 \in \mathbf{F}^3$ as singular locus. In fact, to any polynomial $\varphi \in \mathbf{F}[x, y, z]$, one naturally associates a Poisson bracket with the formula:

$$\{\cdot\,,\cdot\}_\varphi := \frac{\partial\varphi}{\partial x}\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z} + \frac{\partial\varphi}{\partial y}\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial x} + \frac{\partial\varphi}{\partial z}\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}.$$

It is the skew-symmetric biderivation that corresponds to $\mathsf{d}\varphi$ under the natural isomorphism $\Omega^1(\mathcal{A}) \simeq \mathfrak{X}^2(\mathcal{A})$ and the fact that it satisfies the Jacoby identity corresponds to the fact that $\mathsf{d}^2\varphi = 0$. The singular locus of this Poisson structure is the affine variety

$$\left\{\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial y} = \frac{\partial\varphi}{\partial z} = 0\right\} \tag{1.4}$$

and, under some hypotheses, it will be $0 \in \mathbf{F}^3$. In fact, we suppose φ to be a (weight) homogeneous polynomial, that implies that the singular locus (1.4) of the bracket $\{\cdot\,,\cdot\}_\varphi$ coincides with the singularities of the surface

$$\mathcal{F}_\varphi : \{\varphi = 0\} \subseteq \mathbf{F}^3.$$

If φ is a (weight) homogeneous polynomial such that the surface $\mathcal{F}_\varphi$ has isolated singularities (in fact, it then has only one isolated singularity), then we will rather say that φ is *(weight) homogeneous with an isolated singularity*. Now, saying that $\varphi \in \mathbf{F}[x, y, z]$ is a (weight) homogeneous polynomial with an isolated singularity implies that the singular locus of $\mathcal{F}_\varphi$, i.e., the singular locus (1.4) of $\{\cdot\,,\cdot\}_\varphi$, consists of the origin $0 \in \mathbf{F}^3$.

We then determine the Poisson cohomology of the affine Poisson variety $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$, equipped with its algebra of regular functions $\mathcal{A} = \mathbf{F}[x, y, z]$, where $\varphi \in \mathbf{F}[x, y, z]$ is a weight homogeneous polynomial with an isolated singularity. To do this, we write the Poisson coboundary operator, associated to the Poisson variety $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$ with the help of the natural identifications $\mathfrak{X}^0(\mathcal{A}) \simeq \mathcal{A}$, $\mathfrak{X}^1(\mathcal{A}) \simeq \mathcal{A}^3$, $\mathfrak{X}^2(\mathcal{A}) \simeq \mathcal{A}^3$ and $\mathfrak{X}^3(\mathcal{A}) \simeq \mathcal{A}$. We obtain

$$\begin{aligned} &\delta^0(F) = \vec{\nabla}F \times \vec{\nabla}\varphi, \qquad \text{for } F \in \mathcal{A} \simeq \mathfrak{X}^0(\mathcal{A}),\\ &\delta^1(\vec{F}) = -\vec{\nabla}(\vec{F}\cdot\vec{\nabla}\varphi) + \mathrm{Div}(\vec{F})\vec{\nabla}\varphi, \qquad \text{for } \vec{F} \in \mathcal{A}^3 \simeq \mathfrak{X}^1(\mathcal{A}),\\ &\delta^2(\vec{F}) = -\vec{\nabla}\varphi\cdot(\vec{\nabla}\times\vec{F}), \qquad \text{for } \vec{F} \in \mathcal{A}^3 \simeq \mathfrak{X}^2(\mathcal{A}), \end{aligned} \tag{1.5}$$

where $\cdot$ and $\times$ denote respectively the inner and the cross products in $\mathcal{A}^3$, while $\vec{\nabla}$, $\vec{\nabla}\times$ and Div denote respectively the gradient, the curl and the divergence operators. To develop a first idea about our results, one may think of φ as a homogeneous polynomial, of degree denoted by $\varpi(\varphi)$, assuming that its three partial derivatives have only one common zero that is the origin. This implies that

$$\mathcal{A}_{sing}(\varphi) := \frac{\mathbf{F}[x,y,z]}{\left\langle \dfrac{\partial\varphi}{\partial x}, \dfrac{\partial\varphi}{\partial y}, \dfrac{\partial\varphi}{\partial z} \right\rangle}$$

is a finite-dimensional $\mathbf{F}$-vector space. Its dimension is the so-called *Milnor number* μ (see [43]). This space gives information about the (isolated) singularity of the surface $\mathcal{F}_\varphi$ (like multiplicity, see also [14]) as it is exactly the algebra of regular functions on this singularity. We will show that it plays also an important role in the Poisson cohomology of the Poisson variety $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$, so that the Poisson cohomology of $\{\cdot\,,\cdot\}_\varphi$ is closely related to the type of the singularity of $\mathcal{F}_\varphi$.

We consider a family $u_0 = 1, u_1, \dots, u_{\mu-1}$ of homogeneous elements of $\mathcal{A} = \mathbf{F}[x,y,z]$, whose images in $\mathcal{A}_{sing}(\varphi)$ give a $\mathbf{F}$-basis of this $\mathbf{F}$-vector space. The Poisson cohomology spaces of the Poisson variety $(\mathcal{A}, \{\cdot\,,\cdot\}_\varphi)$ are denoted by $H^k(\mathcal{A},\varphi)$.

The algebra of Casimirs of the Poisson algebra $(\mathcal{A}, \{\cdot\,,\cdot\}_\varphi)$ is given in Proposition 3.11 and is simply the algebra generated by φ, that is to say $\mathrm{Cas}(\mathcal{A},\varphi) = H^0(\mathcal{A},\varphi) \simeq \bigoplus_{i\in\mathbf{N}} \mathbf{F}\varphi^i$. In Proposition 3.14, we see that the first Poisson cohomology space of $\mathcal{A}$ is equal to zero if the degree of φ, $\varpi(\varphi)$, is not equal to 3 and otherwise $H^1(\mathcal{A},\varphi)$ is the $\mathrm{Cas}(\mathcal{A},\varphi)$-module given by

$$H^1(\mathcal{A},\varphi) \simeq \mathrm{Cas}(\mathcal{A},\varphi)\,\vec{e}, \qquad \text{if } \varpi(\varphi) = 3,$$

where $\vec{e} := (x,y,z)$ corresponds to the Euler derivation $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$. Notice that the cubic polynomials play a special role here; in the weight homogeneous case, this role is played by the polynomials of degree equal to the sum of the weights of the three variables x, y, z and they correspond to a Poisson structure $\{\cdot\,,\cdot\}_\varphi$ of weighted degree equal to zero. Moreover, with Proposition 3.19, we see that the case $\varpi(\varphi) = 3$ is also the unique case where the biderivation $\{\cdot\,,\cdot\}_\varphi$ is not an exact Poisson structure, i.e. $\{\cdot\,,\cdot\}_\varphi$, which is a 2-cocycle of the Poisson cohomology of $(\mathcal{A}, \{\cdot\,,\cdot\}_\varphi)$, is not a 2-coboundary (see [32]). Proposition 3.19 affirms indeed that the second Poisson cohomology space is exactly

$$\begin{aligned} H^2(\mathcal{A},\varphi) \simeq \bigoplus_{\substack{j\geq 1\\ \varpi(u_j)\neq\varpi(\varphi)-3}} \mathrm{Cas}(\mathcal{A},\varphi)\vec{\nabla}u_j \oplus \bigoplus_{\varpi(u_j)=\varpi(\varphi)-3} \mathrm{Cas}(\mathcal{A},\varphi)u_j\vec{\nabla}\varphi \\ \oplus \bigoplus_{\substack{j\geq 1\\ \varpi(u_j)=\varpi(\varphi)-3}} \mathbf{F}\vec{\nabla}u_j. \end{aligned}$$

Notice that the u_j, and so the singularity of φ, i.e., of the surface $\mathcal{F}_\varphi$ (as the u_j give a basis of the $\mathbf{F}$-vector space $\mathcal{A}_{sing}(\varphi)$) appear in the second Poisson cohomology space $H^2(\mathcal{A},\varphi)$. The writing of $H^2(\mathcal{A},\varphi)$ has been obtained from the third Poisson cohomology space, which is determined in Proposition 3.16, and is the free $\mathrm{Cas}(\mathcal{A},\varphi)$-module generated by the algebra of regular functions on the singularity of $\mathcal{F}_\varphi$:

$$H^3(\mathcal{A},\varphi) \simeq \mathrm{Cas}(\mathcal{A},\varphi) \otimes_{\mathbf{F}} \mathcal{A}_{sing}(\varphi).$$

Notice that $H^2(\mathcal{A},\varphi)$ is not always a free module over the algebra of Casimirs, unlike the other Poisson cohomology spaces of $(\mathcal{A}, \{\cdot\,,\cdot\}_\varphi)$.

In dimension two (class (1))

We point out that the dimension three is the smallest one in which there is a real Jacobi condition. Indeed, in the two-dimensional case, the Jacobi identity is always trivially satisfied. In particular, any skew-symmetric biderivation of $\mathbf{F}[x,y]$ is a Poisson structure on $\mathbf{F}^2$. Moreover, any polynomial $\psi \in \mathbf{F}[x,y]$ leads to a Poisson structure $\{\cdot\,,\cdot\}^\psi := \psi \, \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ and, conversely, every skew-symmetric biderivation or Poisson bracket on $\mathbf{F}^2$ is of this form.

In the class (1) and in dimension two, we compute explicit basis of the Poisson cohomology spaces of weight homogeneous Poisson brackets on $\mathbf{F}^2$, i.e., for Poisson structures given by $\{\cdot\,,\cdot\}^\psi$, where the polynomial $\psi \in \mathbf{F}[x,y]$ is supposed to be weight homogeneous and square free. The methods that we use in this case are inspired by those of determination of the Poisson cohomology in dimension three and are very close to the methods that appear in [45]. It is shown in [54], that the singular locus of the Poisson bracket $\{\cdot\,,\cdot\}^\psi$, namely the curve

$$\mathit{\Gamma}_\psi : \{\psi = 0\}$$

and, more precisely, information about the singularity of $\mathit{\Gamma}_\psi$ appears in the dimensions of the Poisson cohomology spaces. In fact, we will see (in Chapter 4) that the algebra of regular functions on the singularity of $\mathit{\Gamma}_\psi$, given by

$$\mathcal{A}_{sing}(\psi) := \frac{\mathbf{F}[x,y]}{\left\langle \dfrac{\partial\psi}{\partial x}, \dfrac{\partial\psi}{\partial y} \right\rangle},$$

appears in the Poisson cohomology space, as, for example, the second Poisson cohomology space of the Poisson variety $(\mathbf{F}^2, \{\cdot\,,\cdot\}^\psi)$, when ψ is a homogeneous polynomial of degree d, is given, in Proposition 4.11, by

$$H^2(\mathbf{F}^2,\psi) \simeq \mathbf{F}_{d-2}[x,y]\;\psi \;\oplus\; \mathcal{A}_{sing}(\psi),$$

where $\mathbf{F}_{d-2}[x,y]$ is the $\mathbf{F}$-vector space of all the homogeneous polynomials of $\mathbf{F}[x,y]$, of degree $d-2$. We also observe in Proposition 4.10 that the first Poisson cohomology space of $(\mathbf{F}^2, \{\cdot\,,\cdot\}^\psi)$ is

$$H^1(\mathbf{F}^2,\psi) \simeq \mathbf{F}_{d-2}[x,y]\;\vec{e} \oplus \mathbf{F}\vec{\mathcal{H}}_\psi,$$

where $\vec{\mathcal{H}}_\psi = \dfrac{\partial\psi}{\partial y}\dfrac{\partial}{\partial x} - \dfrac{\partial\psi}{\partial x}\dfrac{\partial}{\partial y}$ is the so-called modular derivation of $\{\cdot\,,\cdot\}^\psi$.

For singular surfaces in $\mathbf{F}^3$ (class (2))

When φ is a (weight) homogeneous polynomial, the affine Poisson variety defined above $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$ is closely related to an affine Poisson surface of class (2), where the affine surface considered is singular and the Poisson structure is symplectic everywhere except on the singular locus of the surface. Indeed, as φ is a Casimir for $\{\cdot\,,\cdot\}_\varphi$, the bracket $\{\cdot\,,\cdot\}_\varphi$ induces a Poisson bracket on the surface $\mathcal{F}_\varphi : \{\varphi = 0\} \subset \mathbf{F}^3$. The algebra of regular functions on $\mathcal{F}_\varphi$ is $\mathcal{A}_\varphi := \frac{\mathbf{F}[x,y,z]}{\langle\varphi\rangle}$ and the quotient Poisson bracket obtained is denoted by $\{\cdot\,,\cdot\}_{\mathcal{A}_\varphi}$. Under the hypotheses that φ is weight homogeneous and has an isolated singularity, we compute, in Section 4.3, the Poisson cohomology spaces $H^k(\mathcal{A}_\varphi)$ of this singular Poisson surface $(\mathcal{A}_\varphi, \{\cdot\,,\cdot\}_{\mathcal{A}_\varphi})$.

The Casimirs of $(\mathcal{A}_\varphi, \{\cdot\,,\cdot\}_{\mathcal{A}_\varphi})$ are simply the elements of $\mathbf{F}$ and, in the homogeneous case, according to Propositions 4.21 and 4.22, we have:

$$H^1(\mathcal{A}_\varphi) \simeq \bigoplus_{\varpi(u_j)=\varpi(\varphi)-3} \mathbf{F} u_j\, \vec{e}\,, \qquad H^2(\mathcal{A}_\varphi) \simeq \bigoplus_{\varpi(u_j)=\varpi(\varphi)-3} \mathbf{F} u_j \vec{\nabla}\varphi, \tag{1.6}$$

where the u_j still give a $\mathbf{F}$-vector space basis of the algebra $\mathcal{A}_{sing}(\varphi)$ of regular functions on the singularity of $\mathcal{F}_\varphi$.

Since the coboundary operator associated to the Poisson variety $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$ is a (weight) homogeneous operator (see Paragraph 3.2.1), all our arguments remain true if we replace the algebra $\mathcal{A} = \mathbf{F}[x,y,z]$ by the algebra of all formal power series $\bar{\mathcal{A}} := \mathbf{F}[[x,y,z]]$, still equipped with the Poisson structure $\{\cdot\,,\cdot\}_\varphi$, with φ a (weight) homogeneous element of $\mathcal{A}$. It suffices to replace $\mathrm{Cas}(\mathcal{A},\varphi) = \mathbf{F}[\varphi]$ by $\mathrm{Cas}(\bar{\mathcal{A}},\varphi) = \mathbf{F}[[\varphi]]$, the algebra of formal power series in φ. The analogous result holds for the case of dimension two $(\mathbf{F}^2, \{\cdot\,,\cdot\}^\psi)$.

Poisson homology

In this thesis, we also turn the results on Poisson cohomology to good account to obtain the Poisson homology of the affine Poisson variety $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$ and of the singular Poisson surface $(\mathcal{F}_\varphi, \{\cdot\,,\cdot\}_{\mathcal{A}_\varphi})$.

First, for $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$, using the modular class, we show in Proposition 3.23 that we have isomorphisms

$$H_k(\mathcal{A},\varphi) \simeq H^{3-k}(\mathcal{A},\varphi), \text{ for all } k = 0,1,2,3, \quad \mathcal{A} = \mathbf{F}[x,y,z],$$

where $H_k(\mathcal{A},\varphi)$ denotes the k-th Poisson homology space of the Poisson variety $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$. Notice that, if $\mathbf{F}^3$ is endowed with a Poisson bracket which is not of the form $\{\cdot\,,\cdot\}_\varphi$, $\varphi \in \mathbf{F}[x,y,z]$ (for example, $x\,\{\cdot\,,\cdot\}_\varphi$), then the modular class is not necessarily equal to zero and, in general, this duality does not hold anymore.

We point out that, in [60] and in [41], one can find the computation of the Poisson homology spaces of the Poisson variety $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$ for particular polynomials $\varphi \in \mathbf{F}[x,y,z]$. These cases are particular cases of the Poisson cohomology we compute in this thesis and the method is very similar.

Then, using the results about Poisson cohomology of $(\mathcal{A}, \{\cdot\,,\cdot\}_\varphi)$, we compute the Poisson homology spaces $H_k(\mathcal{A}_\varphi)$ of $\mathcal{F}_\varphi$, equipped with its algebra of regular functions $\mathcal{A}_\varphi$. In this context, one can not define a modular class anymore, and we show that, on the contrary to the case of $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$, there is not a duality between Poisson cohomology and homology spaces. We obtain, in Proposition 4.26,

$$H_0(\mathcal{A}_\varphi) \simeq H_2(\mathcal{A}_\varphi) \simeq \mathcal{A}_{sing}(\varphi)\ , \qquad H_1(\mathcal{A}_\varphi) \simeq \bigoplus_{j=1}^{\mu-1} \mathbf{F}\,\vec{\nabla} u_j.$$

Considering the results of cohomology, given in (1.6), it is clear that the Poisson homology spaces are new invariants of the Poisson surface $(\mathcal{F}_\varphi, \{\cdot\,,\cdot\}_{\mathcal{A}_\varphi})$. We point out that the singular algebra $\mathcal{A}_{sing}(\varphi)$ and hence the singularity of the surface $\mathcal{F}_\varphi$ appears one more time, in the Poisson homology spaces of the singular Poisson surface $(\mathcal{F}_\varphi, \{\cdot\,,\cdot\}_{\mathcal{A}_\varphi})$.

Mathieu-Poisson homology

While for a Poisson variety $(M, \pi = \{\cdot\,,\cdot\})$ and its algebra of regular functions $\mathcal{A}$, the Poisson homology complex consists of the Poisson boundary operator $\partial_k : \Omega^k(\mathcal{A}) \to \Omega^{k-1}(\mathcal{A})$ ($\Omega^\bullet(\mathcal{A})$ denotes the space of all Kähler differentials of $\mathcal{A}$), defined by the formula:

$$\partial_k = \imath_\pi \circ \mathsf{d} - \mathsf{d} \circ \imath_\pi,$$

O. Mathieu has, in [42], introduced a homology with parameter $\tau \in \mathbf{F}$ (we call it the *MP-homology*), for which the corresponding boundary operator $\partial_k^\tau : \Omega^k(\mathcal{A}) \to \Omega^{k-1}(\mathcal{A})$ is rather given by:

$$\partial_k^\tau = (\tau + k)\,\imath_\pi \circ \mathsf{d} - (\tau + k + 1)\,\mathsf{d} \circ \imath_\pi.$$

By using the methods that we have developed for the determination of the "classical" Poisson (co)homology of the affine varieties introduced above, in dimension two: $(\mathbf{F}^2, \{\cdot\,,\cdot\}^\psi)$, $(\mathcal{F}_\varphi, \{\cdot\,,\cdot\}_{\mathcal{A}_\varphi})$ and in dimension three: $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$, we are able to determine the MP-homology of these varieties. In fact, in dimension two, for generic values of τ, we see that the MP-homology is isomorphic to the classical Poisson homology. But, in the case of the Poisson variety of dimension three $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$, there are some differences between these both homologies, viewed in the first and the second homology spaces. In Proposition 5.6, we indeed obtain that, if $\varphi \in \mathbf{F}[x, y, z]$ is a homogeneous polynomial with an isolated singularity and $\tau \in \mathbf{F} \setminus \{-1, -2, -3, -4\}$, then, the first MP-homology space is given by

$$H_1^\tau(\mathcal{A}, \varphi) \simeq \bigoplus_{r \in \mathbf{N}} \bigoplus_{\substack{j=1 \\ \varpi(u_j) \neq \varpi(\varphi) D_\tau^r - 3}}^{\mu-1} \mathbf{F}\varphi^r \vec{\nabla} u_j \oplus \bigoplus_{r \in \mathbf{N}} \bigoplus_{\substack{j=0 \\ \varpi(u_j) = \varpi(\varphi) D_\tau^{r+1} - 3}}^{\mu-1} \mathbf{F}\varphi^r u_j \vec{\nabla} \varphi$$
$$\oplus \bigoplus_{\substack{j=1 \\ \varpi(u_j) = \varpi(\varphi) - 3}}^{\mu-1} \mathbf{F}\vec{\nabla} u_j,$$

where $D_\tau^r := \dfrac{r}{(\tau+2)} + 1 \in \mathbf{F}$. Moreover, in Proposition 5.11, we see that the second MP-homology space is the space

$$H_2^\tau(\mathcal{A},\varphi) \simeq \begin{cases} \{0\} & \text{if for any } r \in \mathbf{N},\ \varpi(\varphi) \neq 3\left(\dfrac{1}{(r+1)\,q_\tau+1}\right), \\ \mathbf{F}\varphi^r\vec{e}, & \text{if } r \in \mathbf{N} \text{ satisfies } \ \varpi(\varphi) = 3\left(\dfrac{1}{(r+1)\,q_\tau+1}\right), \end{cases}$$

where $q_\tau := \dfrac{1}{\tau+2} \in \mathbf{F}$. It follows that, although $H_2(\mathcal{A},\varphi) \simeq H^1(\mathcal{A},\varphi) \simeq \{0\}$, as soon as φ is of degree different from 3, in the case of MP-homology, for any such φ (with $\varpi(\varphi) \neq 3$) and for any $r \in \mathbf{N}$, we can choose a parameter $\tau \in \mathbf{F}$ such that

$$H_2^\tau(\mathcal{A},\varphi) \simeq \mathbf{F}\varphi^r\vec{e} \not\simeq \{0\},$$

i.e., we can "detect" the singularity using H_2^τ (or H_1^τ), upon picking an appropriate value of τ. Notice that, on the contrary of the "classical" Poisson (co)homology spaces, the MP-homology spaces are not modules over the Casimirs and the **F**-vector space $H_2^\tau(\mathcal{A},\varphi)$ is always of finite dimension (equal to zero or one).

An application in deformation theory

As we suggested above, the second and the third Poisson cohomology spaces intervene in deformation theory. In Chapter 6, we use the results obtained in Poisson cohomology to write down all the formal deformations

$$\{\cdot\,,\cdot\}_* = \{\cdot\,,\cdot\}_\varphi + \sum_{k\in\mathbf{N}^*} \pi_k\,\nu^k,$$

of the Poisson bracket $\{\cdot\,,\cdot\}_\varphi$ on $\mathbf{F}^3$, when $\varphi \in \mathbf{F}[x,y,z]$ is a weight homogeneous polynomial, with an isolated singularity, up to equivalence.

The remarkable fact that this can be done explicitly hinges on the following two facts:

1. a term π_{n+1} that integrates the coboundary $\sum\limits_{\substack{i+j=n+1\\ i,j\geq 1}} [\pi_i,\pi_j]_S$ can be explicitly written down for a complete family of n-th order deformations of $\{\cdot\,,\cdot\}_\varphi$, up to equivalence,
2. the freedom of choice for π_{n+1} lies in the second Poisson cohomology space, which we explicitly determined in Section 3.2.

In view of our results on Poisson cohomology, it comes to no surprise that the isolated singularity of φ plays a fundamental role in the deformation properties of the Poisson structure $\{\cdot\,,\cdot\}_\varphi$.

Class (3) = class (1) + (2)

The Poisson structures $\{\cdot,\cdot\}^{\psi}$ on $\mathbf{F}^2$ and $\{\cdot,\cdot\}_{\varphi}$ (or $\{\cdot,\cdot\}_{\mathcal{A}_\varphi}$) on the surface $\mathcal{F}_\varphi$: $\{\varphi = 0\} \subseteq \mathbf{F}^3$ are both particular cases of the general Poisson structure $\psi\{\cdot,\cdot\}_{\varphi}$ on a surface. This general structure is a restriction of the Poisson structure on $\mathbf{F}^3$, given by:

$$\psi\{\cdot,\cdot\}_{\varphi} = \psi\,\frac{\partial\varphi}{\partial x}\,\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z} + \psi\,\frac{\partial\varphi}{\partial y}\,\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial x} + \psi\,\frac{\partial\varphi}{\partial z}\,\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}.$$

Indeed, if one considers the constant polynomial $\psi = 1$, the Poisson bracket $\psi\{\cdot,\cdot\}_{\varphi}$ becomes $\{\cdot,\cdot\}_{\varphi}$ on $\mathbf{F}^3$ and induces $\{\cdot,\cdot\}_{\mathcal{A}_\varphi}$ on the surface $\mathcal{F}_\varphi$, while, if one considers the polynomial $\varphi = z$, then $\mathcal{F}_z \simeq \mathbf{F}^2$, $\mathcal{A}_z = \frac{\mathbf{F}[x,y,z]}{\langle z\rangle} \simeq \mathbf{F}[x,y]$ and the Poisson bracket $\psi\{\cdot,\cdot\}_{\varphi}$ becomes $\psi\,\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} = \{\cdot,\cdot\}^{\psi}$ on $\mathbf{F}^2$.

Ideally, we would like to obtain general results of Poisson (co)homology for the Poisson varieties $(\mathbf{F}^3, \psi\{\cdot,\cdot\}_{\varphi})$ and $(\mathcal{F}_\varphi, \psi\{\cdot,\cdot\}_{\varphi})$, as it could permit us to observe in one hand, the role played by the singularity of the Poisson structure (given by ψ) and in another hand, the role played by the singularity of the surface $\mathcal{F}_\varphi$ (given by φ). Notice that the Casimirs of $\psi\{\cdot,\cdot\}_{\varphi}$ are exactly (if $\psi \neq 0$) the Casimirs of $\{\cdot,\cdot\}_{\varphi}$.

With this idea in mind, we have studied an example in Chapter 7, where ψ is a polynomial of degree 1 ($\psi = x$) and the singularity of φ (or the singularity of the surface $\mathcal{F}_\varphi$) is of type A_n, i.e., $\varphi = \varphi_n = x^{n+1}+y^{n+1}+z^{n+1}$, $n \in \mathbf{N}^*$. We have computed in this case the first and the third Poisson cohomology spaces $H^1(\mathcal{A}; x, \varphi_n)$ and $H^3(\mathcal{A}; x, \varphi_n)$ of the Poisson variety $(\mathbf{F}^3, x\{\cdot,\cdot\}_{\varphi_n})$, equipped with its algebra of regular functions $\mathcal{A} = \mathbf{F}[x,y,z]$. We have obtained, in Propositions 7.3 and 7.6,

$$H^3(\mathcal{A}; x, \varphi_n) \simeq \mathrm{Cas}(\mathcal{A}, \varphi_n) \otimes \frac{\mathbf{F}[x,y,z]}{\langle x^{n+1}, y^n, z^n\rangle},$$

and

$$H^1(\mathcal{A}; x, \varphi_n) \simeq \begin{cases} \mathrm{Cas}(\mathcal{A}, \varphi_n)\left(\vec{\nabla} x \times \vec{\nabla}\varphi_n\right) & \text{if } n \neq 1, \\ \mathrm{Cas}(\mathcal{A}, \varphi_n)\,\vec{e} \oplus \mathrm{Cas}(\mathcal{A}, \varphi_n)\left(\vec{\nabla} x \times \vec{\nabla}\varphi_n\right) & \text{if } n = 1, \end{cases}$$

where $\vec{\nabla} x \times \vec{\nabla}\varphi_n$ corresponds to the modular derivation of the Poisson variety $(\mathbf{F}^3, x\{\cdot,\cdot\}_{\varphi_n})$ and the case $n = 1$ (φ quadratic) corresponds to a Poisson biderivation $x\{\cdot,\cdot\}_{\varphi_n}$ of degree equal to zero.

2

Preliminaries

In this first chapter, our purpose is to introduce the objects that will appear in all this work. There are two words that we will often use: “Poisson” and “singularity”. Around the first one, we need in particular the notion of Poisson variety, Poisson morphisms, Poisson cohomology and homology, while around the second one will appear the isolated singularities, the weight homogeneity and the Koszul complex. We will also specialize these notions to the case of the low dimension (dimension two and three). In fact, these examples of low dimension are important and interesting for the point of view of Poisson cohomology. They correspond to what we study in the further chapters. In this one, as in all this document, $\mathbf{F}$ will be an arbitrary field of characteristic zero.

2.1 Poisson varieties

In this section, we recall the definition of an affine Poisson variety, a Poisson morphism and give some properties of these objects. We will then consider examples in low dimension.

2.1.1 Poisson varieties and their morphisms

An *affine variety* M is an algebraic subset[1] of an affine space $\mathbf{F}^n$, where *algebraic subset* means that M is the zero locus of a family of polynomials (in n variables). Given an algebraic subset $M \subset \mathbf{F}^n$, one considers the ideal $\mathcal{I}$ of $\mathbf{F}[x_1, \dots, x_n]$ that consists of all polynomials, vanishing on M,

$$\mathcal{I} := \{F \in \mathbf{F}[x_1, \dots, x_n] \mid F_{|M} = 0\}.$$

Then $\mathbf{F}[x_1, \dots, x_n]/\mathcal{I}$ becomes a finitely generated algebra, which can be considered as an algebra of functions on M, since the evaluation of elements of $\mathbf{F}[x_1, \dots, x_n]/\mathcal{I}$ at elements of M is well-defined. This algebra of functions on M is denoted by $\mathcal{F}(M)$ and elements of $\mathcal{F}(M)$ are called *regular functions* on M.

[1] Our convention is the French one, for which an affine variety is not necessarily irreducible.

Having at our disposal an algebra of functions on an affine variety M, we can now define the notion of a Poisson structure on M.

Definition 2.1. *Let M be an affine variety and let $\mathcal{A}$ denote its algebra of regular functions. Suppose that $\mathcal{A} = \mathcal{F}(M)$ is equipped with a Lie bracket $\{\cdot\,,\cdot\} : \mathcal{A}\times\mathcal{A} \to \mathcal{A}$, which satisfies the* Leibniz rule *(also called the* derivation property*):*

$$\{FG, H\} = F\{G, H\} + \{F, H\}G, \tag{2.1}$$

for all $F, G, H \in \mathcal{A}$.

Then we say that $(M, \{\cdot\,,\cdot\})$ is an affine Poisson variety*, or simply a* Poisson variety*, but we will also talk about the* Poisson algebra $(\mathcal{A}, \{\cdot\,,\cdot\})$[2]*. The bracket $\{\cdot\,,\cdot\}$ on $\mathcal{A}$ is usually referred to as a* Poisson structure *on M or as a* Poisson bracket *on $\mathcal{A}$.*

Remark 2.2. Suppose that M is an affine variety equipped with its algebra of regular functions $\mathcal{A} = \mathcal{F}(M)$. A linear map $\varphi : \mathcal{A} \to \mathcal{A}$ that satisfies the Leibniz rule (i.e., $\varphi(FG) = F\varphi(G) + \varphi(F)G$, for all $F, G \in \mathcal{A}$) is called a *derivation* of $\mathcal{A}$ (with values in $\mathcal{A}$), while a bilinear map $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ that satisfies the Leibniz rule (the derivation property) in each of its arguments is called a *biderivation* of $\mathcal{A}$ (with values in $\mathcal{A}$). So that, a Poisson bracket on $\mathcal{A}$ is exactly a skew-symmetric biderivation $\{\cdot\,,\cdot\}$ of $\mathcal{A}$, that satisfies the *Jacobi identity*:

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0, \tag{2.2}$$

for all triplets (F, G, H) in $\mathcal{A}^3$.

We denote by $\mathfrak{X}^1(\mathcal{A})$ the $\mathcal{A}$-module of all derivations of $\mathcal{A}$ and by $\mathfrak{X}^2(\mathcal{A})$ the $\mathcal{A}$-module of all skew-symmetric biderivations of $\mathcal{A}$.

Notice that, even if the Lie algebra $(\mathcal{A}, \{\cdot\,,\cdot\})$ is in general infinite-dimensional, the Poisson bracket of arbitrary elements of $\mathcal{A} = \mathcal{F}(M)$ is in the case of an affine variety $M \subset \mathbf{F}^n$ completely determined by the brackets of the generators, an easy consequence of the derivation property (2.1). In the case of a Poisson structure on $\mathbf{F}^n$, we have the following standard formula for the Poisson bracket.

Proposition 2.3. *Let $\{\cdot\,,\cdot\}$ be a Poisson bracket on $\mathcal{A} = \mathbf{F}[x_1, \ldots, x_n]$. For any F, G in $\mathcal{A}$, their Poisson bracket is, in terms of the x_i, given by*

$$\{F, G\} = \sum_{1\le i,j\le n} \{x_i, x_j\} \frac{\partial F}{\partial x_i}\frac{\partial G}{\partial x_j}. \tag{2.3}$$

[2] It is natural to define a *Poisson algebra* as a $\mathbf{F}$-vector space $\mathcal{A}$, equipped with two multiplications, $\cdot$ and $\{\cdot\,,\cdot\}$, such that $(\mathcal{A}, \cdot)$ is an associative commutative $\mathbf{F}$-algebra, $(\mathcal{A}, \{\cdot\,,\cdot\})$ is a Lie algebra and $\{F\cdot G, H\} = F\cdot\{G, H\} + \{F, H\}\cdot G$, for all $F, G, H \in \mathcal{A}$. Thus, the algebra of regular functions on a Poisson variety becomes naturally a Poisson algebra.

Proof. The proof is by the standard argument that two biderivations (or k-derivations, in general) of some associative commutative algebra $\mathcal{A}$ are equal as soon as they agree on a system of generators for $\mathcal{A}$. As we will use this argument several times, let us give the argument here explicitly.

Both sides of (2.3) are bilinear in F and G, so it suffices to show (2.3) when F and G are monomials in $x_1, \dots, x_n$. If G is of degree 0, i.e. $G = 1$, then the right hand side in (2.3) is obviously zero, but also the left hand side, because $\{F, 1 \cdot 1\} = \{F, 1\} 1 + 1 \{F, 1\}$, so that $\{F, 1\} = 0$ for any $F \in \mathcal{A}$; by skew-symmetry, $\{1, G\} = 0$ for any $G \in \mathcal{A}$. Also, the fact that (2.3) holds when F and G are of degree 1, is clear. Suppose now that (2.3) holds when $\deg(F)+\deg(G) \leq n$, for some $n \geq 2$; we show that it holds for all F and G such that $\deg(F)+\deg(G) = n+1$. Let F, G be non-constant monomials, such that $\deg(F)+\deg(G) = n+1$. By skew-symmetry, we may assume that $\deg(F) > 1$. There exist F_1, $F_2 \in \mathcal{A}$, with $\deg(F_1) < \deg(F)$ and $\deg(F_2) < \deg(F)$, such that $F = F_1 F_2$. Since $\{\cdot\, , \cdot\}$ is a biderivation and in view of the recursion hypothesis, we have that

$$\begin{aligned}
\{F, G\} &= \{F_1 F_2, G\} = F_1 \{F_2, G\} + \{F_1, G\} F_2 \\
&= F_1 \sum_{1 \leq i,j \leq n} \{x_i, x_j\} \frac{\partial F_2}{\partial x_i} \frac{\partial G}{\partial x_j} + F_2 \sum_{1 \leq i,j \leq n} \{x_i, x_j\} \frac{\partial F_1}{\partial x_i} \frac{\partial G}{\partial x_j} \\
&= \sum_{1 \leq i,j \leq n} \{x_i, x_j\} \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial x_j}.
\end{aligned}$$

This proves (2.3) for arbitrary polynomials F and G. □

Remark 2.4. We point out that the proof of the previous Proposition 2.3 does not use the fact that $\{\cdot\, , \cdot\}$ satisfies the Jacobi identity, but only the skew-symmetry and the derivation property. So that, any skew-symmetric biderivation $\{\cdot\, , \cdot\}$ of $\mathcal{A}$ (not necessarily Poisson structure) is given by the Formula (2.3). We will see a generalization of this result for the skew-symmetric k-derivations in Proposition 2.15 and 4.14.

With this formula, we can simplify the Jacobi condition in the case of the affine variety $\mathbf{F}^n$ and its algebra of regular functions $\mathcal{A} = \mathbf{F}[x_1, \dots, x_n]$:

Proposition 2.5. *Let us consider $\{\cdot\, , \cdot\}$, a skew-symmetric biderivation of $\mathcal{A} = \mathbf{F}[x_1, \dots, x_n]$. Then $\{\cdot\, , \cdot\}$ satisfies the Jacobi identity (2.2) for any triplet (F, G, H) of elements of $\mathcal{A}$ if and only if it satisfies it for any triplet of generators (x_i, x_j, x_k) of $\mathcal{A} = \mathbf{F}[x_1, \dots, x_n]$.*

Proof. In order to simplify the notations, let us denote by x_{ij} the bracket $\{x_i, x_j\}$, for all $1 \leq i, j \leq n$. For F, G and H in $\mathbf{F}[x_1, \dots, x_n]$, the biderivation property satisfied by $\{\cdot\, , \cdot\}$ and the fact that second order derivatives commute, imply indeed that

$$\begin{aligned}
&\{\{F,G\},H\} + \mathrm{cycl}\,(F,G,H) \\
&\quad = \sum_{i,j,k,l=1}^{n} x_{lk} \frac{\partial x_{ij}}{\partial x_l} \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial x_j} \frac{\partial H}{\partial x_k} + \mathrm{cycl}\,(F,G,H) \\
&\quad = \sum_{i,j,k=1}^{n} \sum_{l=1}^{n} \left(x_{lk} \frac{\partial x_{ij}}{\partial x_l} + \mathrm{cycl}\,(i,j,k) \right) \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial x_j} \frac{\partial H}{\partial x_k},
\end{aligned}$$

so that the Jacobi identity is satisfied for any triplet of elements of $\mathcal{A}$ if and only if, for any triplet of generators (x_i, x_j, x_k):

$$\sum_{l=1}^{n} \left(x_{lk} \frac{\partial x_{ij}}{\partial x_l} + \mathrm{cycl}\,(i,j,k) \right) = 0,$$

which is the Jacobi identity for the triplet (x_i, x_j, x_k). □

We now turn to the notion of a Poisson morphism between two Poisson varieties (defined over the same field $\mathbf{F}$).

Definition 2.6. *Let* $(M_1, \{\cdot\,,\cdot\}_1)$ *and* $(M_2, \{\cdot\,,\cdot\}_2)$ *be two Poisson varieties. A morphism of varieties (also called a regular map)* $\Psi : M_1 \to M_2$ *is called a* Poisson morphism *(or* Poisson map*) if the dual morphism* $\Psi^* : \mathcal{F}(M_2) \to \mathcal{F}(M_1)$ *is a morphism of Lie algebras, i.e.,*

$$\Psi^*\left(\{F,G\}_2\right) = \{\Psi^*(F), \Psi^*(G)\}_1\,,$$

for all $F, G \in \mathcal{F}(M_2)$.

Recall (see e.g. [55, Ch. I.2.3]) that for a morphism of varieties $\Psi : M_1 \to M_2$, the dual morphism Ψ^* is (well-) defined by $\Psi^*(F) := F \circ \Psi$, for all $F \in \mathcal{F}(M_2)$.

It is clear that if $\Psi : M_1 \to M_2$ is a morphism of Poisson varieties and Ψ is an isomorphism of varieties, then $\Psi^{-1} : M_2 \to M_1$ is also a morphism of Poisson varieties. We say then that Ψ is an *isomorphism of Poisson varieties.*

Let us give the definition of a Poisson ideal.

Definition 2.7. *Let* M *be a Poisson variety and let* $\mathcal{A}$ *denote the algebra of regular functions on* M. *An ideal* $\mathcal{I} \subset \mathcal{A}$ *of the associative commutative algebra* $\mathcal{A}$ *is a* Poisson ideal *if*

$$\{\mathcal{I}, \mathcal{A}\} \subset \mathcal{I}. \tag{2.4}$$

In this case, $\mathcal{A}/\mathcal{I}$ *inherits a Poisson bracket from* $\mathcal{A}$.

In view of Definitions 2.1 and 2.6 we get, for a fixed field $\mathbf{F}$, a category whose objects are the Poisson varieties over $\mathbf{F}$ and whose morphisms are the Poisson morphisms.

If $\mathcal{A}$ is the algebra of regular functions on an affine variety, the properties of a Poisson bracket on $\mathcal{A}$ get a geometrical meaning, which leads to many interesting constructions. In this document, we will always deal with affine varieties, see [37] for the case of Poisson manifolds.

Since the Poisson bracket is a biderivation, it leads to a fundamental operation that allows one to associate to elements of $\mathcal{A}$, derivations of $\mathcal{A}$. This operation, which we introduce now, corresponds in the Hamiltonian formulation of classical mechanics to writing the equations of motion for a given Hamiltonian.

Definition 2.8. *Let $(M, \{\cdot\,,\cdot\})$ be a Poisson variety, let $\mathcal{A}$ be its algebra of regular functions and let $H \in \mathcal{A}$. The derivation $\mathcal{X}_H := \{\cdot\,, H\}$ of $\mathcal{A}$ is called a* Hamiltonian derivation *and we call H a* Hamiltonian, *associated to $\mathcal{X}_H$. We write*

$$\mathrm{Ham}\,(\mathcal{A}, \{\cdot\,,\cdot\}) := \{\mathcal{X}_H \mid H \in \mathcal{A}\}$$

for the $\mathbf{F}$-vector space of Hamiltonian derivations of $\mathcal{A}$, so that we have a linear map

$$\begin{aligned} \mathcal{X} : \mathcal{A} &\to \mathrm{Ham}(\mathcal{A}, \{\cdot\,,\cdot\}) \\ H &\mapsto \mathcal{X}_H := \{\cdot\,, H\}\,. \end{aligned} \tag{2.5}$$

An element $H \in \mathcal{A}$ that belongs to the center of the Poisson bracket $\{\cdot\,,\cdot\}$, i.e., such that $\{\cdot\,, H\} = 0$, is called a Casimir *and we denote*

$$\mathrm{Cas}\,(\mathcal{A}, \{\cdot\,,\cdot\}) := \{H \in \mathcal{A} \mid \{\cdot\,, H\} = 0\}$$

for the $\mathbf{F}$-vector space of Casimirs. When no confusion can arise, we write $\mathrm{Ham}(\mathcal{A})$ *for* $\mathrm{Ham}(\mathcal{A}, \{\cdot\,,\cdot\})$ *and* $\mathrm{Cas}(\mathcal{A})$ *for* $\mathrm{Cas}(\mathcal{A}, \{\cdot\,,\cdot\})$.

The defining properties of the Poisson bracket lead to the following proposition.

Proposition 2.9. *Let $(M, \{\cdot\,,\cdot\})$ be an affine Poisson variety and $\mathcal{A} = \mathcal{F}(M)$ be the algebra of regular functions on M. We recall that $\mathfrak{X}^1(\mathcal{A})$ is the space of all derivations of $\mathcal{A}$. The space $\mathfrak{X}^1(\mathcal{A})$, equipped with the commutator of maps $[\cdot\,,\cdot]$ is a Lie algebra.*

(1) $\mathrm{Cas}(\mathcal{A})$ *is a subalgebra of $(\mathcal{A}, \cdot)$;*
(2) $\mathrm{Ham}(\mathcal{A})$ *is* not *an $\mathcal{A}$-module (in general); instead, $\mathcal{X}_{FG} = F\mathcal{X}_G + G\mathcal{X}_F$, for any $F, G \in \mathcal{A}$;*
(3) $\mathrm{Ham}(\mathcal{A})$ *is a* $\mathrm{Cas}(\mathcal{A})$*-module;*
(4) The map $\mathcal{A} \to \mathfrak{X}^1(A)$ which is defined by $H \mapsto -\mathcal{X}_H$ is a morphism of Lie algebras; as a consequence, $\mathrm{Ham}(\mathcal{A})$ *is a Lie subalgebra of $\mathfrak{X}^1(\mathcal{A})$;*
(5) The following is a short exact sequence of Lie algebras

$$0 \longrightarrow \mathrm{Cas}(\mathcal{A}) \longrightarrow \mathcal{A} \xrightarrow{-\mathcal{X}} \mathrm{Ham}(\mathcal{A}) \longrightarrow 0$$

(6) The Lie derivative of $\{\cdot\,,\cdot\}$ with respect to any Hamiltonian derivation is zero; in the manifold case this means that the flow of any Hamiltonian vector field preserves the Poisson structure.

Proof. Properties (1)–(3) follow from the biderivation property (2.1), while (4)–(6) follow from the Jacobi identity (2.2) for $\{\cdot\,,\cdot\}$. □

Remark 2.10. A derivation $\mathcal{V}$ is called a *Poisson derivation* if it preserves the Poisson structure, i.e., if $\mathcal{L}_{\mathcal{V}}\{\cdot\,,\cdot\} = 0$. Proposition 2.9 implies that all Hamiltonian derivations are Poisson derivations. This property will be reformulated in cohomological terms in Section 2.2.

We now give two classical examples of Poisson structures. The first one is the symplectic case and, historically, was the first example of Poisson bracket studied. As we are interesting with the Poisson cohomology, that is still determined in the symplectic case, we will not study this case in the following.

Example 2.11 (Symplectic manifolds). Let (M, ω) be a symplectic manifold. Then, the algebra $\mathcal{F}(M) := C^\infty(M)$ of smooth functions on M, can be equipped with a Poisson bracket. To do that, to any function $F \in \mathcal{F}(M)$, we associate a vector field $\mathcal{X}_F$ with the formula:

$$\mathsf{d}F = \omega(\mathcal{X}_F, \cdot)\,.$$

Then, we define the Poisson bracket of two functions $F, G \in \mathcal{F}(M)$, by putting

$$\{F, G\} := \omega(\mathcal{X}_F, \mathcal{X}_G)\,.$$

This formula defines a Poisson bracket on the algebra $\mathcal{F}(M)$. A. Lichnérowicz, in [40], has shown that the Poisson cohomology (see Section 2.2) of a symplectic variety is isomorphic to the de Rham one.

The second example that we give in this paragraph is the Lie-Poisson case and corresponds to our algebraic context. Some particular cases of the Poisson brackets we will study are Lie-Poisson structures.

Example 2.12 (Lie-Poisson structure). Let us consider a Lie algebra $(\mathfrak{g}, [\cdot\,,\cdot])$, of finite dimension $n \in \mathbf{N}$. The dual of $\mathfrak{g}$ can be identified with $\mathbf{F}[x_1, \ldots, x_n]$ ($\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$) and is naturally a Poisson variety. A natural Poisson bracket is indeed given by the formula (2.3), where each bracket $\{x_i, x_j\}$ is equal to the Lie bracket $[x_i, x_j]$.

2.1.2 Example: dimension two

We consider the affine space of dimension two, $\mathbf{F}^2$, equipped with its algebra of regular functions $\mathcal{A} = \mathbf{F}[x, y]$. Let us determine the Poisson brackets that can be defined on $\mathcal{A}$.

According to the definition 2.1 and Remark 2.2, we have to consider the skew-symmetric biderivations of $\mathcal{A}$. Let $\{\cdot\,,\cdot\}$ be a skew-symmetric biderivation of $\mathcal{A}$, according to Proposition 2.3 and Remark 2.4, it will be totally determined by the bracket of the two generators x and y. We denote by $F \in \mathcal{A} = \mathbf{F}[x, y]$ this polynomial, $F = \{x, y\}$, so that

$$\{\cdot\,,\cdot\} = F\,\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}\,.$$

Then, it remains (see Remark 2.2), for $\{\cdot\,,\cdot\}$ to be a Poisson bracket, to verify the Jacobi condition. But, as we have seen in the Proposition 2.5, it is sufficient to verify the Jacobi identity on the generators. Because there are only two generators and because of the skew-symmetry of $\{\cdot\,,\cdot\}$, the Jacobi identity in dimension two will always be trivially satisfied, so that all skew-symmetric biderivation of $\mathbf{F}[x,y]$ is a Poisson structure on $\mathbf{F}^2$. We thus have a correspondence between the Poisson brackets on $\mathcal{A}$ and the polynomials of $\mathcal{A}$:

$$\begin{array}{ccc} \{\text{Poisson brackets}\} = \mathfrak{X}^2(\mathcal{A}) & \longleftrightarrow & \mathcal{A} = \mathbf{F}[x,y] \\ \{\cdot\,,\cdot\} & \longrightarrow & \{x,y\} \\ F\,\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} & \longleftarrow & F \end{array}$$

2.1.3 Example: dimension three

In this paragraph, the affine variety we consider is the affine space of dimension three, namely $M = \mathbf{F}^3$, with its algebra of regular functions, which is the polynomial algebra $\mathcal{A} = \mathbf{F}[x,y,z]$. Our purpose is to determine the Poisson structures existing on this variety. We point out that the dimension three is the first dimension where one has to verify a real Jacobi identity, as, in dimension two (see Paragraph 2.1.2), the Jacobi is trivially satisfied.

Poisson structures on $\mathbf{F}^3$

Let $\{\cdot\,,\cdot\} : \wedge^2\mathcal{A} \to \mathcal{A}$ be a skew-symmetric biderivation on $\mathcal{A}$. According to Proposition 2.3 and Remark 2.4, it is completely defined by the brackets of the three generators x, y, z, that is to say, by the three polynomials $F_1, F_2, F_3 \in \mathcal{A}$, defined by:

$$\{y,z\} = F_1, \quad \{z,x\} = F_2, \quad \{x,y\} = F_3.$$

Explicitly, we have:

$$\{\cdot\,,\cdot\} = F_1\,\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z} + F_2\,\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial x} + F_3\,\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}.$$

This bracket will be a Poisson bracket on $\mathcal{A}$ if and only if it satisfies the Jacobi identity (see Remark 2.2). Let us write this identity for $\{\cdot\,,\cdot\}$, in terms of F_1, F_2 and F_3. In fact, we have seen in Proposition 2.5 that it suffices to write the Jacobi identity for the generators x, y and z:

$$\begin{aligned} \{\{x,y\},z\} + \{\{z,x\},y\} + \{\{y,z\},x\} \ &= \ \{F_3,z\} + \{F_2,y\} + \{F_1,x\} = \\ \left(F_1\,\frac{\partial F_3}{\partial y} - F_2\,\frac{\partial F_3}{\partial x}\right) &+ \left(F_3\,\frac{\partial F_2}{\partial x} - F_1\,\frac{\partial F_2}{\partial z}\right) + \left(F_2\,\frac{\partial F_1}{\partial z} - F_3\,\frac{\partial F_1}{\partial y}\right) = \\ F_1\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) &+ F_2\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + F_3\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = \\ \vec{F}\cdot(\vec{\nabla}\times\vec{F}), & \end{aligned}$$

where $\vec{F}$ denotes the triplet of polynomials that defines $\{\cdot\,,\cdot\}$, namely,

$$\vec{F} := (F_1, F_2, F_3) \in \mathcal{A}^3,$$

and where $\cdot$ and $\vec{\nabla}\times$ denote respectively the inner product and the curl operator in $\mathcal{A}^3$. In fact, in the following, we will see that it will be convenient to use the notations of the vector calculus of $\mathbf{R}^3$, adapted to $\mathcal{A}^3$. Also, $\vec{\nabla}$ will denote the gradient operator and $\times$ the cross product.

The previous reasoning shows that the Poisson structures on $\mathcal{A} = \mathbf{F}[x, y, z]$ are given by triplets of polynomials $\vec{F} = (F_1, F_2, F_3) \in \mathcal{A}^3$ that satisfy the identity:

$$\vec{F} \cdot (\vec{\nabla} \times \vec{F}) = 0. \tag{2.6}$$

Now, let us consider a Poisson bracket $\{\cdot\,,\cdot\}$ on $\mathbf{F}^3$, given by such a triplet of polynomials $\vec{F} = (F_1, F_2, F_3)$. Let $G, H \in \mathcal{A} = \mathbf{F}[x, y, z]$ be two polynomials and let us compute their bracket $\{G, H\}$:

$$\begin{aligned}\{G,H\} &= \left(F_1\,\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z} + F_2\,\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial x} + F_3\,\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\right)(G,H)\\ &= F_1\,\left(\frac{\partial G}{\partial y}\frac{\partial H}{\partial z} - \frac{\partial H}{\partial y}\frac{\partial G}{\partial z}\right) + F_2\,\left(\frac{\partial G}{\partial z}\frac{\partial H}{\partial x} - \frac{\partial H}{\partial z}\frac{\partial G}{\partial x}\right)\\ &\qquad + F_3\,\left(\frac{\partial G}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial H}{\partial x}\frac{\partial G}{\partial y}\right)\\ &= \vec{F}\cdot\left(\vec{\nabla}G \times \vec{\nabla}H\right).\end{aligned} \tag{2.7}$$

This formula is a useful expression of the Poisson bracket of two polynomials F, G, in dimension three.

Poisson structures associated to polynomials

According to Equation (2.6), one can see that, to any polynomial $\varphi \in \mathcal{A}$, corresponds a Poisson bracket on $\mathcal{A}$, denoted by $\{\cdot\,,\cdot\}_\varphi$ and defined by the triplet $\vec{F} = \vec{\nabla}\varphi := \left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right)$, namely:

$$\{\cdot\,,\cdot\}_\varphi := \frac{\partial\varphi}{\partial x}\,\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z} + \frac{\partial\varphi}{\partial y}\,\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial x} + \frac{\partial\varphi}{\partial z}\,\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}. \tag{2.8}$$

One has indeed $(\vec{\nabla} \times \vec{\nabla}\varphi) = 0$, so that condition (2.6) is necessarily satisfied for any $\vec{F} = \vec{\nabla}\varphi$, with $\varphi \in \mathcal{A}$.

The writing of the Jacobi condition as in (2.6) shows also easily that, if $\{\cdot\,,\cdot\}$ is a Poisson bracket on $\mathcal{A}$ (given by a triplet of polynomials $\vec{F}$) and $\chi \in \mathcal{A}$ is an arbitrary polynomial, then the skew-symmetric biderivation $\chi\{\cdot\,,\cdot\}$ (given by the triplet of polynomials $\chi\vec{F} := (\chi F_1, \chi F_2, \chi F_3) \in \mathcal{A}^3$) is also a Poisson bracket on $\mathcal{A}$. This fact does not happen in general, in other dimensions. In fact, one has the following formula, well-known from vector calculus in $\mathbf{R}^3$:

$$\vec{\nabla} \times (F\vec{G}) = \vec{\nabla}F \times \vec{G} + F(\vec{\nabla} \times \vec{G}), \tag{2.9}$$

for all $F \in \mathcal{A}$ and $\vec{G} \in \mathcal{A}^3$. So that, if $\vec{F} \cdot (\vec{\nabla} \times \vec{F}) = 0$, then:

$$\chi\vec{F} \cdot \left(\vec{\nabla} \times (\chi\vec{F})\right) = \chi\vec{F} \cdot \left(\vec{\nabla}\chi \times \vec{F} + \chi(\vec{\nabla} \times \vec{F})\right) = 0.$$

In particular, for any $\chi, \varphi \in \mathcal{A}$, we have a Poisson bracket on $\mathcal{A}$, given by $\chi\{\cdot\,,\cdot\}_\varphi$, where $\{\cdot\,,\cdot\}_\varphi$ is defined by (2.8).

Poisson structures on surfaces in $\mathbf{F}^3$

Let us consider a Poisson structure on $M = \mathbf{F}^3$, of the form $\chi\{\cdot\,,\cdot\}_\varphi$, where $\chi, \varphi \in \mathcal{A} = \mathbf{F}[x, y, z]$ are arbitrary polynomials.

It is easy to check that φ is a Casimir for this Poisson structure, as, according to Formula (2.7), for any $F \in \mathcal{A}$,

$$\chi\{F, \varphi\}_\varphi = \chi\vec{\nabla}\varphi \cdot \left(\vec{\nabla}F \times \vec{\nabla}\varphi\right) = 0,$$

so that, according to Definition 2.7, the ideal $\langle\varphi\rangle$ is a Poisson ideal of $\mathcal{A}$, equipped with the Poisson bracket $\chi\{\cdot\,,\cdot\}_\varphi$ and the algebra $\mathcal{A}/\langle\varphi\rangle$ inherits a Poisson bracket from $(\mathcal{A}, \chi\{\cdot\,,\cdot\}_\varphi)$. Said differently, the Poisson bracket $\chi\{\cdot\,,\cdot\}_\varphi$ goes to the quotient $\mathcal{A}/\langle\varphi\rangle$ and the map induced is a Poisson bracket on $\mathcal{A}/\langle\varphi\rangle$.

The geometrical meaning of this fact is that the surface defined by the zeros of the polynomial φ, $M_\varphi : \{\varphi = 0\} \subset \mathbf{F}^3$, can be equipped with the Poisson structure induced by $\chi\{\cdot\,,\cdot\}_\varphi$.

2.2 Poisson (co)homology

A Poisson structure on an affine variety M leads in a natural way to cohomology spaces, derived from the multi-derivations of its algebra of regular functions $\mathcal{F}(M)$, and to homology spaces, derived from the Kähler differentials of the algebra $\mathcal{F}(M)$. These spaces give information on the derivations, normal forms, deformations and several invariants of the Poisson structure. In some specific, but important, cases they are related to classically known cohomology spaces, as the ones that appear in de Rham cohomology or in Lie algebra cohomology. In general, Poisson cohomology is finer, but is also more difficult to compute, as we will see. In this chapter, we construct the various complexes that lead to these (co)homologies and we show that, under certain conditions, the Poisson cohomology and Poisson homology spaces are dual to each other.

2.2.1 Cohomology

We define Poisson cohomology for a Poisson variety $(M, \{\cdot\,,\cdot\})$, also referred to as the Poisson cohomology of $\mathcal{A} = \mathcal{F}(M)$, the algebra of regular functions over M. The Poisson structure $\{\cdot\,,\cdot\}$ on M will also be denoted by π, so that $\pi(F, G) = \{F, G\}$ for $F, G \in \mathcal{A}$.

The cochains: skew-symmetric multi-derivations

For $k \in \mathbf{N}$, the space of k-cochains of the Poisson cohomology complex is $\mathfrak{X}^k(\mathcal{A})$, the vector space of skew-symmetric k-derivations of $\mathcal{A}$. Let us define these objects, which are generalizations of the skew-symmetric biderivations (see Remark 2.2) and give some properties that will be useful in this section. For more details, see the book [37].

Definition 2.13. *Let $\mathcal{A}$ be an associative commutative algebra over $\mathbf{F}$ (for example, the algebra of regular functions over an affine variety). A skew-symmetric k-linear map $P \in \mathrm{Hom}_{\mathbf{F}}(\wedge^k\mathcal{A}, \mathcal{A})$ is called a skew-symmetric* k-derivation *of $\mathcal{A}$ (with values in $\mathcal{A}$), when it is a derivation in each of its arguments, i.e.,*

$$P(FG, F_2, \dots, F_k) = F\,P(G, F_2, \dots, F_k) + G\,P(F, F_2, \dots, F_k), \tag{2.10}$$

for arbitrary elements $F, G, F_2, \dots, F_k$ of $\mathcal{A}$.

In the following, we will rather use square brackets for the arguments of a k-derivation, so that $P(F_1, \dots, F_k)$ will be denoted by $P[F_1, \dots, F_k]$, for P a skew-symmetric k-derivation and $F_1, \dots, F_k \in \mathcal{A}$.

The $\mathcal{A}$-module of skew-symmetric k-derivations on $\mathcal{A}$ is denoted by $\mathfrak{X}^k(\mathcal{A})$ and we introduce the graded $\mathcal{A}$-module

$$\mathfrak{X}^\bullet(\mathcal{A}) := \bigoplus_{k\in\mathbf{N}} \mathfrak{X}^k(\mathcal{A}) \subset \bigoplus_{k\in\mathbf{N}} \mathrm{Hom}_{\mathbf{F}}(\wedge^k\mathcal{A}, \mathcal{A}),$$

whose elements are called skew-symmetric *multi-derivations*. By convention, the first term in this sum, $\mathfrak{X}^0(\mathcal{A})$, is just $\mathcal{A}$, and $\mathfrak{X}^k(\mathcal{A}) := \{0\}$ for $k < 0$.

Remark 2.14. If $\mathcal{A}$ is a finitely generated algebra $\mathcal{A} = \mathbf{F}[x_1, \dots, x_n]/\mathcal{I}$, then any skew-symmetric k-derivation is equal to zero as soon as $k > n$.

As for the Poisson bracket in Proposition 2.3, we will give a formula for any skew-symmetric k-derivation of $\mathbf{F}[x_1, \dots, x_n]$.

Proposition 2.15. *Let P be a skew-symmetric k-derivation of the polynomial algebra $\mathbf{F}[x_1, \dots, x_n]$. Then P is totally defined by its values on the generators $x_1, \dots, x_n$ and we have, explicitly:*

$$P[F_1, \dots, F_k] = \sum_{1\le i_1,\dots,i_k\le n} P[x_{i_1}, \dots, x_{i_k}]\,\frac{\partial F_1}{\partial x_{i_1}} \cdots \frac{\partial F_k}{\partial x_{i_k}},$$

for all $F_1, \dots, F_k \in \mathcal{A}$. In other words, we have the equality:

$$P = \sum_{1 \le i_1 < \dots < i_k \le n} P[x_{i_1}, \dots, x_{i_k}] \frac{\partial}{\partial x_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x_{i_k}}. \tag{2.11}$$

Proof. The proof is the one of Proposition 2.3, adapted to a k-derivation. □

Remark 2.16. In a geometrical case, when M is a manifold, the skew-symmetric k-derivations of $\mathcal{F}(M) = C^\infty(M)$ correspond to the k-vector fields of M.

The Poisson cohomology complex

Let $(M, \pi = \{\cdot\,,\cdot\})$ be an affine variety and let $\mathcal{A}$ be its algebra of regular functions. We recall that we will also denote by π the Poisson structure $\{\cdot\,,\cdot\}$ on M. The *Poisson coboundary operator* associated to $(\mathcal{A}, \{\cdot\,,\cdot\})$ or $(M, \{\cdot\,,\cdot\})$, $\delta_\pi : \mathfrak{X}^\bullet(\mathcal{A}) \to \mathfrak{X}^{\bullet+1}(\mathcal{A})$, is the graded $\mathbf{F}$-linear map, defined for $Q \in \mathfrak{X}^q(\mathcal{A})$ by

$$\begin{aligned} \delta^q_\pi(Q)[F_0, \dots, F_q] &:= \sum_{i=0}^{q} (-1)^i \left\{ F_i, Q[F_0, \dots, \widehat{F_i}, \dots, F_q] \right\} \\ &+ \sum_{0 \le i < j \le q} (-1)^{i+j} Q \left[\{F_i, F_j\}, F_0, \dots, \widehat{F_i}, \dots, \widehat{F_j}, \dots, F_q \right], \end{aligned} \tag{2.12}$$

for all $F_0, \dots, F_q \in \mathcal{A}$. When no confusion can arise, we will sometimes denote the Poisson coboundary operator by δ_π. By a direct computation, one checks that $\delta^q_\pi(Q)$ is indeed a skew-symmetric multi-derivation and that $\delta^q_\pi \circ \delta^{q-1}_\pi = 0$ (See also the next paragraph, where we will introduce a bracket on the skew-symmetric multi-derivations, that will be related to the Poisson coboundary operator).

We so have a complex, the *Poisson cohomology complex* of $\mathcal{A}$,

$$\cdots \longrightarrow \mathfrak{X}^{q-1}(\mathcal{A}) \xrightarrow{\delta^{q-1}_\pi} \mathfrak{X}^q(\mathcal{A}) \xrightarrow{\delta^q_\pi} \mathfrak{X}^{q+1}(\mathcal{A}) \xrightarrow{\delta^{q+1}_\pi} \cdots \tag{2.13}$$

The elements of $Z^q(\mathcal{A}, \pi) := \operatorname{Ker} \delta^q_\pi$ are called *Poisson q-cocycles* while the elements of $B^q(\mathcal{A}, \pi) := \operatorname{Im} \delta^{q-1}_\pi$ are called *Poisson q-coboundaries.* Elements of the *q-th Poisson cohomology space* are Poisson q-cocycles modulo Poisson q-coboundaries,

$$H^q(\mathcal{A}, \pi) := \operatorname{Ker} \delta^q_\pi / \operatorname{Im} \delta^{q-1}_\pi,$$

for $q \ge 1$ and $H^0(\mathcal{A}, \pi) := \operatorname{Ker} \delta^0_\pi$. The graded vector space

$$H^\bullet(\mathcal{A}, \pi) := \bigoplus_{q \in \mathbf{N}} H^q(\mathcal{A}, \pi),$$

is called the *Poisson cohomology* of $(M, \{\cdot\,,\cdot\})$, also referred to as the Poisson cohomology of $(\mathcal{A}, \{\cdot\,,\cdot\})$.

Remark 2.17. It is easy to point out that the Poisson coboundary operator δ^q_π commutes with the multiplication by a Casimir of the Poisson variety (See Definition 2.8), so that, each q-th Poisson cohomology space is a $\mathrm{Cas}(\mathcal{A},\pi)$-module.

For small q, the Poisson coboundary operator δ^q_π has a natural interpretation, which yields a natural interpretation for the Poisson cohomology spaces $H^q(\mathcal{A},\pi)$; see Chapter 6 for applications in deformation theory. One easily reads off from (2.12) that for $F \in \mathfrak{X}^0(\mathcal{A}) = \mathcal{A}$ and for $\mathcal{V} \in \mathfrak{X}^1(\mathcal{A})$

$$\delta^0_\pi(F) = \mathcal{X}_F, \qquad \delta^1_\pi(\mathcal{V}) = -\mathcal{L}_{\mathcal{V}}\pi, \tag{2.14}$$

where we recall that $\mathcal{X}_F$ is the Hamiltonian derivation, associated to $F \in \mathcal{A}$ (see Definition 2.8), and $\mathcal{L}_{\mathcal{V}}$ denotes the Lie derivative with respect to $\mathcal{V}$. Also, for $Q \in \mathfrak{X}^2(\mathcal{A})$ and $F, G, H \in \mathcal{A}$,

$$\delta^2_\pi(Q)[F,G,H] = \{F, Q[G,H]\} + Q[F,\{G,H\}] + \mathrm{cycl}(F,G,H).$$

It follows from (2.14) that Poisson 0-cocycles are Casimirs,

$$H^0(\mathcal{A},\pi) = \mathrm{Cas}(\mathcal{A},\pi),$$

Poisson 1-cocycles are Poisson derivations (see Remark 2.10), while Poisson 1-coboundaries are Hamiltonian derivations. Denoting the space of all Poisson derivations of $\mathcal{A}$ by $\mathcal{P}(\mathcal{A},\pi)$, we have that

$$H^1(\mathcal{A},\pi) = \frac{\mathcal{P}(\mathcal{A},\pi)}{\mathrm{Ham}(\mathcal{A},\pi)}.$$

Poisson 2-cocycles $Q \in \mathfrak{X}^2(\mathcal{A})$ are skew-symmetric biderivations that satisfy

$$\{F, Q[G,H]\} + Q[F,\{G,H\}] + \mathrm{cycl}(F,G,H) = 0.$$

We say that they are *compatible* with $\{\cdot\,,\cdot\}$ (see the next paragraph to have a general definition of compatible skew-symmetric multi-derivations); Poisson 2-coboundaries are biderivations, obtained as a Lie derivative of the Poisson structure. It follows that

$$H^2(\mathcal{A},\pi) = \frac{\text{skew-symmetric biderivations compatible with } \pi}{\text{Lie derivatives of } \pi}.$$

$H^2(\mathcal{A},\pi)$ and $H^3(\mathcal{A},\pi)$ will show up naturally in deformation theory, see Chapter 6.

Remark 2.18. Contrary to what we will see for Poisson homology in next Paragraph 2.2.2, Poisson cohomology is not a functor: a homomorphism $\varphi : M_1 \to M_2$ between two Poisson varieties does not lead in general to a homomorphism between the spaces of all multi-derivations of $\mathcal{F}(M_1)$ and $\mathcal{F}(M_2)$, nor between their corresponding Poisson cohomology groups. In the case of manifolds for example, it is well known that vector fields cannot be transferred, in either direction, by a smooth map, that is not a diffeomorphism.

The Schouten bracket

Let $\mathcal{A}$ be an associative commutative algebra. In this paragraph, we give the definition of a bracket on the skew-symmetric multi-derivations of $\mathcal{A}$, the Schouten bracket, that is related to the Poisson coboundary operator and permits one to prove easily some properties of it, when $\mathcal{A}$ is equipped with a Poisson bracket.

The set of all (p,q)-shuffles[3] is denoted by $S_{p,q}$. For a shuffle $\sigma \in S_{p,q}$, we denote the signature of σ (as a permutation) by $\epsilon(\sigma)$. It is also convenient to define $S_{p,-1} := \emptyset$ and $S_{-1,q} := \emptyset$, for $p,q \in \mathbf{N}$.

Let us recall the *Schouten bracket*, which is a product on multi-derivations, based on the composition of multi-derivations. It is a family of maps

$$[\cdot\,,\cdot]_S : \mathfrak{X}^p(\mathcal{A}) \times \mathfrak{X}^q(\mathcal{A}) \to \mathfrak{X}^{p+q-1}(\mathcal{A}),$$

for $p,q \in \mathbf{N}$. It is defined, for $P \in \mathfrak{X}^p(\mathcal{A})$, $Q \in \mathfrak{X}^q(\mathcal{A})$ and $F_1,\ldots,F_{p+q-1} \in \mathcal{A}$, by

$$\begin{aligned} &[P,Q]_S[F_1,\ldots,F_{p+q-1}] \\ &\quad = \sum_{\sigma \in S_{q,p-1}} \epsilon(\sigma) P\left[Q[F_{\sigma(1)},\ldots,F_{\sigma(q)}], F_{\sigma(q+1)},\ldots,F_{\sigma(q+p-1)}\right] \\ &\quad - (-1)^{(p-1)(q-1)} \sum_{\sigma \in S_{p,q-1}} \epsilon(\sigma) Q\left[P[F_{\sigma(1)},\ldots,F_{\sigma(p)}], F_{\sigma(p+1)},\ldots,F_{\sigma(p+q-1)}\right]. \end{aligned} \tag{2.15}$$

A priori, we have only that $[P,Q]_S \in \mathrm{Hom}(\mathcal{A}^{\otimes(p+q-1)},\mathcal{A})$, but it is easy to check that indeed $[P,Q]_S \in \mathfrak{X}^{p+q-1}(\mathcal{A})$.

The Schouten bracket can be seen as a generalization of many classical elementary operations on functions, derivations and multi-derivations. First, let $Q := F \in \mathcal{A}$ and $P \in \mathfrak{X}^p(\mathcal{A})$. Then $[P,F]_S = \imath_F P$, where $\imath_F$ is the graded $\mathcal{A}$-linear map $\imath_F : \mathfrak{X}^\bullet(\mathcal{A}) \to \mathfrak{X}^{\bullet-1}(\mathcal{A})$, defined by taking F as the first element on which the multi-derivation is evaluated: for $P \in \mathfrak{X}^p(\mathcal{A})$, the $(p-1)$-derivation $\imath_F P$ is defined by

$$\imath_F P[F_2,\ldots,F_p] := P[F,F_2\ldots,F_p], \tag{2.16}$$

for all $F_2,\ldots,F_p \in \mathcal{A}$. Second, let $P := \mathcal{V} \in \mathfrak{X}^1(\mathcal{A})$ and $Q \in \mathfrak{X}^q(\mathcal{A})$. Then $[\mathcal{V},Q]_S = \mathcal{L}_\mathcal{V} Q$, the Lie derivative of Q with respect to $\mathcal{V}$. Third, the Schouten bracket of two skew-symmetric biderivations $P,Q \in \mathfrak{X}^2(\mathcal{A})$ is given by

$$[P,Q]_S[F_1,F_2,F_3] = P[Q[F_1,F_2],F_3] + Q[P[F_1,F_2],F_3] + \mathrm{cycl}(F_1,F_2,F_3),$$

so that, for $P \in \mathfrak{X}^2(\mathcal{A})$, one has:

$$[P,P]_S[F_1,F_2,F_3] = 2\Big(P[P[F_1,F_2],F_3] + \mathrm{cycl}(F_1,F_2,F_3)\Big).$$

This leads to the following result.

[3] For $p,q \in \mathbf{N}$, a (p,q)-*shuffle* is a permutation σ of the set $\{1,\ldots,p+q\}$, such that $\sigma(1) < \ldots < \sigma(p)$ and $\sigma(p+1) < \ldots < \sigma(p+q)$.

Proposition 2.19. *If P is a skew-symmetric biderivation of $\mathcal{A}$, i.e., $P \in \mathfrak{X}^2(\mathcal{A})$, then P defines a Poisson bracket on $\mathcal{A}$ (that is to say, satisfies the Jacobi identity) if and only if $[P, P]_S = 0$.*

More generally, when $P, Q \in \mathfrak{X}^2(\mathcal{A})$ and $[P, Q]_S = 0$, one says that P and Q are *compatible*. It follows that two Poisson brackets P and Q are compatible if and only if their sum (or any linear combination with non-zero coefficients) is a Poisson bracket.

A different way to establish the properties of the Poisson coboundary operator δ_π is to relate it to the Schouten bracket. In fact, the explicit Formula (2.15) for the Schouten bracket $[\cdot\,,\cdot]_S$, implies that

$$\delta_\pi^q(Q) = -[Q, \pi]_S, \text{ for } Q \in \mathfrak{X}^q(\mathcal{A}), \tag{2.17}$$

thus, $\delta_\pi^q(Q)$ is a skew-symmetric $(q+1)$-derivation. To see that $\delta_\pi^q \circ \delta_\pi^{q-1} = 0$, we consider Q' a skew-symmetric $(q-1)$-derivation of $\mathcal{A}$ and we write down the graded Jacobi identity of the Schouten bracket[4] for the triplet (Q', π, π),

$$(-1)^q \left[Q', [\pi, \pi]_S\right]_S - \left[\pi, [Q', \pi]_S\right]_S + (-1)^q \left[\pi, [\pi, Q']_S\right]_S = 0. \tag{2.18}$$

Since $[\pi, \pi]_S = 0$ (as π is a Poisson bracket, see Proposition 2.19) and since $[Q', \pi]_S = -(-1)^q [\pi, Q']_S$ (because of the graded anti-commutativity of $[\cdot\,,\cdot]_S$), the identity (2.18) reduces to $\left[\pi, [\pi, Q']_S\right]_S = 0$, for any $Q' \in \mathfrak{X}^{q-1}(\mathcal{A})$, showing that $\delta_\pi^q \circ \delta_\pi^{q-1} = 0$.

2.2.2 Homology

A Poisson bracket on an affine variety also leads to homology spaces. In special cases they are isomorphic to the Poisson cohomology spaces, but in general they define new invariants for a Poisson variety. The Poisson homology spaces have less direct applications than the Poisson cohomology spaces, but have the advantage of being (slightly) easier to compute and have better functorial properties, as we will see. Let $(M, \{\cdot\,,\cdot\})$ be a Poisson variety and let $\mathcal{A} = \mathcal{F}(M)$ be its algebra of regular functions.

The chains: Kähler differentials

The k-chains that define the Poisson homology complex are the Kähler differentials on $\mathcal{A}$. For a good definition of the Kähler differentials over an arbitrary associative commutative algebra $\mathcal{A}$, see [20].

We recall that the $\mathcal{A}$-module of *Kähler differentials* of $\mathcal{A}$ (over $\mathbf{F}$) is denoted by $\Omega^1(\mathcal{A})$ and the graded $\mathcal{A}$-module $\Omega^p(\mathcal{A}) := \bigwedge^p \Omega^1(\mathcal{A})$ is the module of all *Kähler*

[4] $(-1)^{(p-1)(r-1)}[P, [Q, R]_S]_S + \text{cycl}(P, Q, R) = 0$, for P (respectively Q, R) a skew-symmetric p-derivation (respectively skew-symmetric q, r-derivations) (graded Jacobi identity).

p-differentials. As a vector space, resp. as an $\mathcal{A}$-module, $\Omega^p(\mathcal{A})$ is generated by elements of the form $F\mathsf{d}F_1 \wedge \ldots \wedge \mathsf{d}F_p$, resp. of the form $\mathsf{d}F_1 \wedge \ldots \wedge \mathsf{d}F_p$, where F, $F_i \in \mathcal{A}$ for $1 \leq i \leq p$. We denote by $\Omega^\bullet(\mathcal{A}) = \bigoplus_{p\in\mathbf{N}} \Omega^p(\mathcal{A})$ the space of all *Kähler differentials*.

The differential $\mathsf{d} : \mathcal{A} \to \Omega^1(\mathcal{A})$ extends to a graded $\mathbf{F}$-linear map

$$\mathsf{d} : \Omega^\bullet(\mathcal{A}) \to \Omega^{\bullet+1}(\mathcal{A}),$$

by putting

$$\mathsf{d}(F\,\mathsf{d}F_1 \wedge \cdots \wedge \mathsf{d}F_p) := \mathsf{d}F \wedge \mathsf{d}F_1 \wedge \cdots \wedge \mathsf{d}F_p, \tag{2.19}$$

for $F, F_1, \ldots, F_p \in \mathcal{A}$, where $p \in \mathbf{N}$. It is called the *de Rham differential*. It is a graded derivation, of degree 1, of $(\Omega^\bullet(\mathcal{A}), \wedge)$, such that $\mathsf{d} \circ \mathsf{d} = 0$. The resulting complex is the so-called *de Rham complex* and its cohomology is the *de Rham cohomology*.

The Poisson homology complex

To simplify the notations, we will also denote by π the Poisson bracket $\pi = \{\cdot\,,\cdot\}$ on M. The *Poisson boundary operator*, also called the Brylinsky or Koszul differential and denoted by $\partial^\pi : \Omega^\bullet(\mathcal{A}) \to \Omega^{\bullet-1}(\mathcal{A})$, is given by

$$\begin{aligned}\partial^\pi_k(F_0\,\mathsf{d}F_1 \wedge \cdots \wedge \mathsf{d}F_k) &= \sum_{i=1}^{k}(-1)^{i+1}\{F_0, F_i\}\;\mathsf{d}F_1 \wedge \cdots \wedge \widehat{\mathsf{d}F_i} \wedge \ldots \wedge \mathsf{d}F_k \\ &+ \sum_{1\leq i<j\leq k}(-1)^{i+j}F_0\,\mathsf{d}\{F_i, F_j\} \wedge \mathsf{d}F_1 \wedge \cdots \wedge \widehat{\mathsf{d}F_i} \wedge \cdots \wedge \widehat{\mathsf{d}F_j} \wedge \cdots \wedge \mathsf{d}F_k,\end{aligned}$$

where $F_0, \ldots, F_k \in \mathcal{A}$. When no confusion can arise, we will sometimes denote by ∂^π the Poisson boundary operator.

As for δ^k_π, one can check, by a direct computation, that ∂^π_k is well-defined and that it is a boundary operator, $\partial^\pi_k \circ \partial^\pi_{k+1} = 0$ (See also the end of this section to see another writing of the Poisson boundary operator, that will help us to verify its properties).

We then have a complex

$$\cdots \longrightarrow \Omega^{k+1}(\mathcal{A}) \xrightarrow{\partial^\pi_{k+1}} \Omega^k(\mathcal{A}) \xrightarrow{\partial^\pi_k} \Omega^{k-1}(\mathcal{A}) \xrightarrow{\partial^\pi_{k-1}} \cdots \tag{2.20}$$

whose homology is called the *Poisson homology* of $(M, \{\cdot\,,\cdot\})$, also referred to as the Poisson homology of $(\mathcal{A}, \{\cdot\,,\cdot\})$. Namely, we define the *k-th Poisson homology space*

$$H_k(\mathcal{A}, \pi) := \operatorname{Ker}\partial^\pi_k / \operatorname{Im}\partial^\pi_{k+1},$$

and we call elements of $Z_k(\mathcal{A}, \pi) := \operatorname{Ker}\partial^\pi_k$ *Poisson k-cycles* and elements of $B_k(\mathcal{A}, \pi) := \operatorname{Im}\partial^\pi_{k+1}$ *Poisson k-boundaries*.

Remark 2.20. As for the Poisson cohomology, the boundary operator δ_k^π commutes with the multiplication by a Casimir, so that the Poisson homology spaces are modules over the spaces of the Casimirs.

For example, $H_0(\mathcal{A},\pi)$ is the abelianization of $\mathcal{A}$: since $\operatorname{Ker}\partial_0^\pi = \mathcal{A}$ while $\operatorname{Im}\partial_1^\pi = \{\mathcal{A},\mathcal{A}\}$, we have that

$$H_0(\mathcal{A},\pi) = \frac{\mathcal{A}}{\{\mathcal{A},\mathcal{A}\}}. \tag{2.21}$$

For the higher Poisson homology spaces, a simple interpretation is not known.

Remark 2.21. Unlike Poisson cohomology, Poisson homology is a (contravariant) functor. Indeed, for any morphism $\varphi : M_1 \to \mathcal{M}_2$ of affine varieties, the dual map $\varphi^* : \mathcal{A}_2 = \mathcal{F}(M_2) \to \mathcal{A}_1 = \mathcal{F}(M_1)$ extends to a degree zero map $\Omega^\bullet(\varphi^*) : \Omega^\bullet(\mathcal{A}_2) \to \Omega^\bullet(\mathcal{A}_1)$, which commutes with d. If M_1 and M_2 are equipped with Poisson structures π_1 and π_2 and φ is a Poisson morphism, then $\Omega^\bullet(\varphi^*)\circ\imath_{\pi_2} = \imath_{\pi_1}\circ\Omega^\bullet(\varphi^*)$, so that $\Omega^\bullet(\varphi^*)\circ\partial^{\pi_2} = \partial^{\pi_1}\circ\Omega^\bullet(\varphi^*)$. Thus, there is an induced map $H_\bullet(\varphi^*) : H_\bullet(\mathcal{A}_2,\pi_2) \to H_\bullet(\mathcal{A}_1,\pi_1)$, which is a homomorphism of graded vector spaces. Clearly, $H_\bullet$ has all properties that are required for a (contravariant) functor.

The inner product

For any associative commutative algebra $\mathcal{A}$ and any $p \in \mathbf{N}$, there is a natural action of $\mathfrak{X}^p(\mathcal{A})$ on $\Omega^\bullet(\mathcal{A})$. It is the graded $\mathcal{A}$-linear map

$$\imath_P : \Omega^\bullet(\mathcal{A}) \to \Omega^{\bullet-p}(\mathcal{A}),$$

which is defined for $P \in \mathfrak{X}^p(\mathcal{A})$ and $\omega = \mathsf{d}F_1\wedge\ldots\wedge\mathsf{d}F_n \in \Omega^n(\mathcal{A})$, by

$$\imath_P(\mathsf{d}F_1\wedge\ldots\wedge\mathsf{d}F_n) := \sum_{\sigma\in S_{p,n-p}} \epsilon(\sigma)P[F_{\sigma(1)},\ldots,F_{\sigma(p)}]\,\mathsf{d}F_{\sigma(p+1)}\wedge\ldots\wedge\mathsf{d}F_{\sigma(n)} \tag{2.22}$$

when $n \geq p$, and otherwise $\imath_P\omega := 0$. Notice that, in particular, $\imath_F\omega = F\omega$ for any $F \in \mathcal{A}$ and $\omega \in \Omega^\bullet(\mathcal{A})$. In the following proposition, we establish the main properties of $\imath$.

Proposition 2.22. *Let $\mathcal{A}$ be an associative, commutative algebra. The following identities hold, for $P \in \mathfrak{X}^p(\mathcal{A})$ and $Q \in \mathfrak{X}^q(\mathcal{A})$:*

(1) $\imath_P\circ\imath_Q = \imath_{Q\wedge P}$;
(2) $[\imath_P,\imath_Q] = 0$;
(3) For any $F \in \mathcal{A}$ and $\omega \in \Omega^\bullet(\mathcal{A})$:

$$\imath_P(\mathsf{d}F\wedge\omega) = \imath_{(\imath_F P)}\omega + (-1)^p\mathsf{d}F\wedge\imath_P\omega. \tag{2.23}$$

In the following proposition, we express the Schouten bracket in terms of the de Rham differential, with the help of the inner product.

Proposition 2.23. *If P and Q are two skew-symmetric multi-derivations of $\mathcal{A}$, then*

$$[[\imath_P, \mathsf{d}], \imath_Q] = \imath_{[P,Q]_S}, \tag{2.24}$$

where $[\cdot\,,\cdot]$ is the graded commutator of graded maps on $\Omega^\bullet(\mathcal{A})$. This formula is called Cartan's Formula.

For the proof of this proposition, see [37]. The main properties of ∂^π follow now from the following proposition.

Proposition 2.24. *Let $(M, \{\cdot\,,\cdot\})$ be a Poisson variety and $\mathcal{A}$ be its algebra of regular functions. We will also denote by π the Poisson bracket $\pi = \{\cdot\,,\cdot\}$. Then the boundary operator ∂^π, defined above, is given by the commutator*

$$\partial^\pi = [\imath_\pi, \mathsf{d}] = \imath_\pi \circ \mathsf{d} - \mathsf{d} \circ \imath_\pi. \tag{2.25}$$

It commutes (in the graded sense) with d and with $\imath_\pi$,

$$[\partial^\pi, \mathsf{d}] = \partial^\pi \circ \mathsf{d} + \mathsf{d} \circ \partial^\pi = 0, \tag{2.26}$$

$$[\partial^\pi, \imath_\pi] = \partial^\pi \circ \imath_\pi - \imath_\pi \circ \partial^\pi = 0, \tag{2.27}$$

and satisfies $\partial^\pi \circ \partial^\pi = 0$ (it is a boundary operator).

Proof. For the biderivation π the inner product $\imath_\pi$ is, according to (2.22), explicitly given by

$$\begin{aligned}
&\imath_\pi(F_0\, \mathsf{d}F_1 \wedge \ldots \wedge \mathsf{d}F_k) \\
&\quad = \sum_{1 \le i < j \le k} (-1)^{i+j+1} F_0 \{F_i, F_j\}\, \mathsf{d}F_1 \wedge \ldots \wedge \widehat{\mathsf{d}F_i} \ldots \widehat{\mathsf{d}F_j} \wedge \ldots \wedge \mathsf{d}F_k,
\end{aligned}$$

so that

$$\begin{aligned}
&\imath_\pi \circ \mathsf{d}(F_0\, \mathsf{d}F_1 \wedge \ldots \wedge \mathsf{d}F_k) \\
&\quad = \sum_{0 \le i < j \le k} (-1)^{i+j+1} \{F_i, F_j\}\, \mathsf{d}F_0 \wedge \ldots \wedge \widehat{\mathsf{d}F_i} \ldots \widehat{\mathsf{d}F_j} \wedge \ldots \wedge \mathsf{d}F_k,
\end{aligned}$$

and

$$\begin{aligned}
&\mathsf{d} \circ \imath_\pi(F_0\, \mathsf{d}F_1 \wedge \ldots \wedge \mathsf{d}F_k) \\
&\quad = \sum_{1 \le i < j \le k} (-1)^{i+j+1} \{F_i, F_j\}\, \mathsf{d}F_0 \wedge \ldots \wedge \widehat{\mathsf{d}F_i} \ldots \widehat{\mathsf{d}F_j} \wedge \ldots \wedge \mathsf{d}F_k, \\
&\quad + \sum_{1 \le i < j \le k} (-1)^{i+j+1} F_0\, \mathsf{d}\{F_i, F_j\} \wedge \mathsf{d}F_1 \wedge \ldots \wedge \widehat{\mathsf{d}F_i} \ldots \widehat{\mathsf{d}F_j} \wedge \ldots \wedge \mathsf{d}F_k.
\end{aligned}$$

Taking the difference of these two formulas leads at once to $\partial^\pi = \imath_\pi \circ \mathsf{d} - \mathsf{d} \circ \imath_\pi = [\imath_\pi, \mathsf{d}]$. Moreover,

$$\partial^\pi \circ \mathsf{d} + \mathsf{d} \circ \partial^\pi = \imath_\pi \circ \mathsf{d} \circ \mathsf{d} - \mathsf{d} \circ \imath_\pi \circ \mathsf{d} + \mathsf{d} \circ \imath_\pi \circ \mathsf{d} - \mathsf{d} \circ \mathsf{d} \circ \imath_\pi = 0,$$

because $\mathsf{d} \circ \mathsf{d} = 0$. The Cartan's formula (2.24) imply that

$$[\partial^\pi, \imath_\pi] = [[\imath_\pi, \mathsf{d}], \imath_\pi] = \imath_{[\pi,\pi]_S} = 0,$$

where we used in the last equality that π is a Poisson structure (See Proposition 2.19). We have then obtained (2.26) and (2.27). Using these two properties, we have $\imath_\pi \circ \mathsf{d} \circ \partial^\pi = -\partial^\pi \circ \imath_\pi \circ \mathsf{d}$ and $\mathsf{d} \circ \imath_\pi \circ \partial^\pi = -\partial^\pi \circ \mathsf{d} \circ \imath_\pi$, so that:

$$\begin{aligned}\partial^\pi \circ \partial^\pi &= \imath_\pi \circ \mathsf{d} \circ \partial^\pi - \mathsf{d} \circ \imath_\pi \circ \partial^\pi \\ &= -\partial^\pi \circ \imath_\pi \circ \mathsf{d} + \partial^\pi \circ \mathsf{d} \circ \imath_\pi \\ &= -\partial^\pi \circ \partial^\pi\end{aligned}$$

which shows that $\partial^\pi \circ \partial^\pi = 0$, i.e., ∂^π is a boundary operator. □

2.2.3 Modular class and duality

The modular class of a Poisson variety $(M, \pi = \{\cdot\,,\cdot\})$ is a Poisson cohomology class (element of $H^1(\mathcal{A}, \pi)$, where $\mathcal{A} = \mathcal{F}(M)$) that is canonically associated to the Poisson bracket. We will not give the description in the most general case and restrict ourselves to the case of the affine space $\mathbf{F}^n$, equipped with its algebra of polynomial functions $\mathcal{A} = \mathbf{F}[x_1, \ldots, x_n]$.

In the case of an orientable real Poisson manifold, equipped with its algebra of real smooth functions or in the case of $\mathbf{C}^n$, equipped with its algebra of holomorphic functions, the definitions and arguments would have to be adapted. Indeed, in our case, the volume form is unique (up to a constant), while this is not true in general.

Our goal is to find a sufficient condition for the Poisson homology to be the dual of the Poisson cohomology. It is well-known that for a symplectic manifold (M, ω), it is true (See [39] and [40]). Thus, Poisson structures for which cohomology and homology are dual may be considered as a generalization of symplectic structure.

The modular derivation

We consider the affine variety $\mathbf{F}^n$, equipped with its polynomial algebra of functions $\mathcal{A} := \mathbf{F}[x_1, \ldots, x_n]$ and with a Poisson structure $\{\cdot\,,\cdot\}$, also denoted by π. We denote by λ the standard volume form $\lambda = \mathsf{d}x_1 \wedge \ldots \wedge \mathsf{d}x_n$.

In this case, we can define the so-called *star operator* $\star$, which is a family of isomorphisms $\star : \mathfrak{X}^k(\mathcal{A}) \to \Omega^{n-k}(\mathcal{A})$, defined for $Q \in \mathfrak{X}^k(\mathcal{A})$ by

$$\star Q = \imath_Q \lambda. \tag{2.28}$$

This operator allows us to transpose the de Rham differential of the Kähler differential forms of $\mathcal{A}$, to the multi-derivations of $\mathcal{A}$. We then obtain the *divergence*. The divergence (with respect to λ) is the graded $\mathbf{F}$-linear map (of degree -1), $\mathrm{Div} : \mathfrak{X}^{\bullet}(\mathcal{A}) \to \mathfrak{X}^{\bullet-1}(\mathcal{A})$, which makes the following diagram commute

$$\begin{array}{ccc} \mathfrak{X}^{\bullet}(\mathcal{A}) & \xrightarrow{\ \star\ } & \Omega^{n-\bullet}(\mathcal{A}) \\ {\scriptstyle \mathrm{Div}}\downarrow & & \downarrow{\scriptstyle \mathsf{d}} \\ \mathfrak{X}^{\bullet-1}(\mathcal{A}) & \xrightarrow[\ \star\ ]{} & \Omega^{n-\bullet+1}(\mathcal{A}) \end{array} \tag{2.29}$$

For example, in the case of $\mathbf{R}^n$, the star of a derivation $\mathcal{V} = \sum_{i=1}^{n} F_i \, \partial/\partial x_i$ is given by

$$\star \mathcal{V} = \sum_{i=1}^{n} (-1)^{i-1} F_i \mathsf{d}x_1 \wedge \ldots \wedge \widehat{\mathsf{d}x_i} \wedge \ldots \wedge \mathsf{d}x_n,$$

so that $\mathrm{Div}(\mathcal{V})$ is given by the classical formula

$$\mathrm{Div}(\mathcal{V}) = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}.$$

Definition 2.25. *The derivation* $\mathrm{Div}(\pi) \in \mathfrak{X}^1(\mathcal{A})$ *is called the* modular derivation *of* $(\mathcal{A}, \pi = \{\cdot\,,\cdot\})$ *(or* $(M, \{\cdot\,,\cdot\})$*).*

We have the following result:

Proposition 2.26. *Let* $\mathbf{F}^n$ *be the affine space of dimension* $n \in \mathbf{N}^*$*, equipped with its algebra of polynomial functions* $\mathbf{F}[x_1, \ldots, x_n]$ *and its canonical volume form* $\lambda = \mathsf{d}x_1 \wedge \ldots \wedge \mathsf{d}x_n$*. The modular derivation* $\mathrm{Div}(\pi)$ *is a Poisson derivation, i.e., it is a Poisson* 1*-cocycle.*

Proof. We first point out that

$$\partial^{\pi}(\lambda) = [\imath_{\pi}, \mathsf{d}](\lambda) = -\mathsf{d} \circ \imath_{\pi}(\lambda) = -\mathsf{d} \star \pi = - \star \mathrm{Div}(\pi) = -\imath_{\mathrm{Div}(\pi)}\lambda. \tag{2.30}$$

Then, using this formula and the Cartan formula (2.24), we have

$$\begin{aligned} \star\, \delta^1_{\pi}(\mathrm{Div}(\pi)) &= \star[\pi, \mathrm{Div}(\pi)]_S = \imath_{[\pi, \mathrm{Div}(\pi)]_S}(\lambda) \\ &= \left[[\imath_{\pi}, \mathsf{d}]\, , \imath_{\mathrm{Div}(\pi)} \right] (\lambda) \\ &= \left[\partial^{\pi}, \imath_{\mathrm{Div}(\pi)} \right] (\lambda) \\ &= \partial^{\pi} \circ \imath_{\mathrm{Div}(\pi)}(\lambda) + \imath_{\mathrm{Div}(\pi)} \circ \partial^{\pi}(\lambda) \\ &= -\partial^{\pi} \circ \partial^{\pi}(\lambda) - \imath_{\mathrm{Div}(\pi)} \circ \imath_{\mathrm{Div}(\pi)}(\lambda) \\ &= 0, \end{aligned}$$

because, $\partial^{\pi} \circ \partial^{\pi} = 0$ and, as $\mathrm{Div}(\pi) \in \mathfrak{X}^1(\mathcal{A})$, we have $\imath_{\mathrm{Div}(\pi)} \circ \imath_{\mathrm{Div}(\pi)} = 0$. So that, $\delta^1_{\pi}(\mathrm{Div}(\pi)) = 0$ and $\mathrm{Div}(\pi)$ is a Poisson 1-cocycle. □

Definition 2.27. *Let $\mathbf{F}^n$ be the affine space of dimension $n \in \mathbf{N}^*$, equipped with a Poisson structure $\pi = \{\cdot\,,\cdot\}$, with its algebra of polynomial functions $\mathcal{A} := \mathbf{F}[x_1, \dots, x_n]$ and its canonical volume form $\lambda = \mathsf{d}x_1 \wedge \dots \wedge \mathsf{d}x_n$. If the modular derivation* $\mathrm{Div}(\pi)$ *is equal to zero, then we say that* $(\mathbf{F}^n, \{\cdot\,,\cdot\})$ *(or $(\mathcal{A}, \{\cdot\,,\cdot\})$) is* unimodular*.*

Unimodular Poisson varieties

We now prove that, if $(\mathbf{F}^n, \pi = \{\cdot\,,\cdot\})$ is unimodular, then its Poisson homology and cohomology spaces are isomorphic. To do this, we first show how the Poisson boundary and coboundary operators are related via the modular derivation $\mathrm{Div}(\pi)$. To do this, we specialize Cartan's Formula (2.24), valid for $P \in \mathfrak{X}^p(M)$ and $Q \in \mathfrak{X}^q(M)$, to the case $P = \pi$, giving

$$\partial^\pi \circ \imath_Q - (-1)^q \imath_Q \circ \partial^\pi = (-1)^{q-1} \imath_{\delta_\pi(Q)}. \tag{2.31}$$

If we apply (2.31) to λ and use Formula (2.30), then we find

$$\partial^\pi(\star Q) + (-1)^q \imath_Q \circ \imath_{\mathrm{Div}(\pi)} \lambda = (-1)^{q-1} \star (\delta_\pi(Q)),$$

which we also write, using $\imath_Q \circ \imath_P = \imath_{P \wedge Q}$, valid for $P, Q \in \mathfrak{X}^\bullet(M)$ (see Proposition 2.22), as follows:

$$- \star (\mathrm{Div}(\pi) \wedge Q) = \star \delta_\pi(Q) + (-1)^q \partial^\pi(\star Q). \tag{2.32}$$

The latter formula says that the modular derivation $\mathrm{Div}(\pi)$ measures the non-commutativity of the following diagram

$$\begin{array}{ccc} \mathfrak{X}^\bullet(M) & \xrightarrow{\ \star\ } & \Omega^{n-\bullet}(M) \\ \downarrow{\scriptstyle \delta_\pi} & \overset{\star\,\mathrm{Div}(\pi)\wedge}{\searrow} & \downarrow{\scriptstyle \partial^\pi} \\ \mathfrak{X}^{\bullet+1}(M) & \xrightarrow[\ \star\ ]{} & \Omega^{n-1-\bullet}(M) \end{array} \tag{2.33}$$

This leads at once to the following duality theorem. (see also [65] and [31]).

Proposition 2.28. *If the affine space $\mathbf{F}^n$, equipped with its polynomial algebra $\mathcal{A} = \mathbf{F}[x_1, \dots, x_n]$ and with a Poisson structure $\pi = \{\cdot\,,\cdot\}$ is unimodular, then its Poisson cohomology spaces and Poisson homology spaces are isomorphic, in the following sense:*

$$H^k(\mathcal{A}, \pi) \simeq H_{n-k}(\mathcal{A}, \pi), \qquad 0 \le k \le n. \tag{2.34}$$

Proof. As $\mathrm{Div}(\pi) = 0$, (2.32) implies that the isomorphism $\star : \mathfrak{X}^\bullet(\mathcal{A}) \to \Omega^{n-\bullet}(\mathcal{A})$ is a bijection between the Poisson cocycles (resp. coboundaries) of $(\mathcal{A}, \pi = \{\cdot\,,\cdot\})$ and the Poisson cycles (resp. boundaries) of $(\mathcal{A}, \pi = \{\cdot\,,\cdot\})$. It follows that $\star$ induces isomorphisms, as indicated in (2.34). $\square$

Remark 2.29 (the case of real orientable manifold).

If $(M, \{\cdot\,,\cdot\})$ is a real orientable Poisson manifold, equipped with its algebra of real smooth functions $\mathcal{A} = C^\infty(M)$, then we can do a reasoning analogous to the one we have done in the case of the affine space $\mathbf{F}^n$, but adapted to the fact that the volume form is not unique anymore. The key fact is that the Poisson cohomology class of the modular derivation does not depend on the choice of the volume form and is then called the *modular class* of $(M, \{\cdot\,,\cdot\})$. If this modular class is trivial (M is *unimodular*), one can check that we have the same result than Proposition 2.28.

Remark 2.30. For an arbitrary affine variety, we can not, in general, define the modular class, as there is no volume form.

2.3 Weight homogeneity, isolated singularities and Koszul complex

In the next chapters, we will study the Poisson cohomology associated to a weight homogeneous polynomial, with isolated singularities and we will, in this part, recall the definitions and some results about theses notions.

2.3.1 Weight homogeneous multi-derivations

The main part of this work will concern weight homogeneous Poisson structures in low dimensions. Let us recall the definition of a weight homogeneous Poisson bracket, and, more generally, the definition of a weight homogeneous skew-symmetric multi-derivation. Let $\mathcal{A} = \mathbf{F}[x_1, \ldots, x_n]$ be a polynomial $\mathbf{F}$-algebra.

A non-zero multi-derivation $P \in \mathfrak{X}^\bullet(\mathcal{A})$ is said to be *weight homogeneous* of (*weighted*) degree $r \in \mathbf{Z}$, if there exist positive integers $\varpi_1, \varpi_2, \ldots, \varpi_n \in \mathbf{N}^*$ (the *weights* of the variables $x_1, x_2, \ldots, x_n$), without a common divisor, such that

$$\mathcal{L}_{\vec{e}_\varpi}[P] = rP,$$

where $\mathcal{L}_{\vec{e}_\varpi}$ is the Lie derivative with respect to the (weight homogeneous) Euler derivation

$$\vec{e}_\varpi := \varpi_1\, x_1 \frac{\partial}{\partial x_1} + \varpi_2\, x_2 \frac{\partial}{\partial x_2} + \cdots + \varpi_n\, x_n \frac{\partial}{\partial x_n}. \tag{2.35}$$

The degree of a weight homogeneous multi-derivation $P \in \mathfrak{X}^\bullet(\mathcal{A})$ is also denoted by $\varpi(P) \in \mathbf{Z}$. By convention, the zero k-derivation is weight homogeneous of degree $-\infty$.

For the particular case of $\mathfrak{X}^0(\mathcal{A}) \simeq \mathcal{A}$, the definition of weight homogeneity becomes: a polynomial $F \in \mathcal{A} = \mathbf{F}[x_1, \ldots, x_n]$ is weight homogeneous of (weighted) degree $\varpi(F)$, if there exist positive integers $\varpi_1, \varpi_2, \ldots, \varpi_n \in \mathbf{N}^*$, without a common divisor, such that

$$\varpi_1\, x_1\, \frac{\partial F}{\partial x_1} + \varpi_2\, x_2\, \frac{\partial F}{\partial x_2} + \cdots + \varpi_n\, x_n\, \frac{\partial F}{\partial x_n} = \varpi(F)\, F. \tag{2.36}$$

This equality is the so-called *Euler Formula* and is equivalent to

$$F(\lambda^{\varpi_1} x_1, \ldots, \lambda^{\varpi_n} x_n) = \lambda^{\varpi(F)} F(x_1, \ldots, x_n), \text{ for all } \lambda \in \mathbf{F}.$$

Let us point out that, in the particular case $\varpi_1 = \cdots = \varpi_n = 1$, the weight homogeneity of F is exactly the "classical" homogeneity of F.

Remark 2.31. According to the definition of weight homogeneity, if $x_1, \ldots, x_n$ are already equipped with weights $\varpi_1, \ldots, \varpi_n$, then, for $1 \leq i_1 < \cdots < i_k \leq n$, $\dfrac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \dfrac{\partial}{\partial x_{i_k}}$ is a weight homogeneous skew-symmetric k-derivation of $\mathcal{A}$, of weighted degree:

$$\varpi\Big(\frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_k}}\Big) = -(\varpi_{i_1} + \cdots + \varpi_{i_k}). \tag{2.37}$$

Notice that the degree of a non-zero k-derivation may be negative for $k > 0$.

Fixing weights $\varpi_1, \ldots, \varpi_n \in \mathbf{N}^*$ for the variables $x_1, \ldots, x_n$, it is clear that $\mathcal{A} = \bigoplus_{i \in \mathbf{N}} \mathcal{A}_i$, where $\mathcal{A}_0 = \mathbf{F}$ and for $i \in \mathbf{N}^*$, $\mathcal{A}_i$ is the $\mathbf{F}$-vector space generated by all weight homogeneous polynomials of degree i. More generally, denoting by $\mathfrak{X}^k(\mathcal{A})_i$ ($k \in \mathbf{N}, i \in \mathbf{Z}$) the $\mathbf{F}$-vector space generated by all weight homogeneous skew-symmetric k-derivations of degree i, $\mathfrak{X}^k(\mathcal{A})_i := \{P \in \mathfrak{X}^k(\mathcal{A}) \mid \varpi(P) = i\} \cup \{0\}$, we have also

$$\mathfrak{X}^k(\mathcal{A}) = \bigoplus_{i \in \mathbf{Z}} \mathfrak{X}^k(\mathcal{A})_i\,, \qquad \text{for all } k \in \mathbf{N}.$$

2.3.2 Isolated singularities in $\mathbf{F}[x_1, \ldots, x_n]$

Let us consider a hypersurface $\mathcal{F}$ in $\mathbf{C}^n$, defined by the zeros of a polynomial $\Phi \in \mathbf{C}[x_1, \ldots, x_n]$,

$$\mathcal{F} : \{\Phi = 0\}.$$

We suppose that $\Phi(0) = 0$, that is to say, we suppose that the origin is a point of $\mathcal{F}$. Then one says that $\mathcal{F}$ (or the polynomial Φ) has a singularity at the origin, if all the n first order partial derivatives of Φ vanishe at this point.

Then one says that this singularity is isolated if there exists a neighborhood of the origin, containing no other singularity of Φ.

In this case, the $\mathbf{C}$-vector space

$$\mathbf{C}[x_1, \ldots, x_n] / \Big\langle \frac{\partial \Phi}{\partial x_1}, \ldots, \frac{\partial \Phi}{\partial x_n} \Big\rangle$$

is of finite dimension (see [14]).

If Φ is a weight homogeneous polynomial of degree $\varpi(\Phi)$ greater than each ϖ_i $(1 \leq i \leq n)$, where ϖ_i is the weight of the variable x_i, then the origin is always a singularity of Φ. This singularity is isolated, if and only if, the origin is the only common zero of the partial derivatives of Φ.

Algebraically ($\mathbf{C}$ becomes $\mathbf{F}$, an arbitrary field of characteristic zero), we say that a weight homogeneous polynomial Φ of $\mathcal{A} = \mathbf{F}[x_1, \ldots, x_n]$ has an *isolated singularity* (at the origin) if and only if

$$\left\{\frac{\partial \Phi}{\partial x_1} = \cdots = \frac{\partial \Phi}{\partial x_n} = 0\right\}_{\bar{\mathbf{F}}} = \{0\},$$

where the zeros of the partial derivatives of Φ are considered in $\bar{\mathbf{F}}^n$, $\bar{\mathbf{F}}$ denoting the algebraic closure of $\mathbf{F}$. That is equivalent to say that

$$\mathcal{A}_{sing}(\Phi) := \mathbf{F}[x_1, \ldots, x_n]/\left\langle \frac{\partial \Phi}{\partial x_1}, \ldots, \frac{\partial \Phi}{\partial x_n}\right\rangle \tag{2.38}$$

is finite-dimensional, as a $\mathbf{F}$-vector space. The dimension of this $\mathbf{F}$-vector space is then called the *Milnor number* of the singular point.

When no confusion can arise, we will denote by $\mathcal{A}_{sing}$ this $\mathbf{F}$-vector space. The notation $\mathcal{A}_{sing}$ (or $\mathcal{A}_{sing}(\Phi)$) is justified as follows. By definition, $\mathcal{A}_{sing}(\Phi)$ is exactly the $\mathbf{F}$-algebra of regular functions over the singularity of Φ and is called the *singular algebra* (or the *Milnor algebra*) associated to Φ.

In the case $n = 3$ and $\varphi \in \mathbf{F}[x, y, z]$, this affine variety is the singular locus of the Poisson structure $\{\cdot\,,\cdot\}_\varphi$, defined in (2.8). This singular algebra $\mathcal{A}_{sing}(\varphi)$ associated to φ will play an important role in the Poisson cohomology of the Poisson algebras $(\mathcal{A}, \{\cdot\,,\cdot\}_\varphi)$ and $(\mathcal{A}_\varphi, \{\cdot\,,\cdot\}_{\mathcal{A}_\varphi})$ (See chapter 3 and Section 4.3).

2.3.3 The Koszul complex associated to a polynomial

The Koszul complex (see [20] for more details) is a homological tool that extends the following case. For R a ring and $x \in R$, one can consider the complex:

$$K(x): \quad 0 \longrightarrow R \stackrel{x}{\longrightarrow} R$$

where the map $R \stackrel{x}{\to} R$ is the multiplication by x. If x is not a zero divisor, then the cohomology of this complex is trivial:

$$H^0(K(x)) = \{y \in R \mid xy = 0\} = 0.$$

As well as the notion of regular sequence extends the notion of non zero divisor (see Paragraph 3.1.3), the Koszul complexes extend the case $K(x)$.

A simple way to construct Koszul complexes in general is to consider a R-module N and the exterior algebra $\wedge^\bullet N$. Then, for $x \in N$, one considers the complex

$$K(x): \quad 0 \longrightarrow R \xrightarrow{\wedge x} N \xrightarrow{\wedge x} \wedge^2 N \xrightarrow{\wedge x} \cdots$$

In the context that will interest us, we specialize this definition to

$$\begin{gathered} R = \mathcal{A} = \mathbf{F}[x_1, \ldots, x_n], \\ N = \Omega^1(\mathcal{A}), \quad \wedge^\bullet N = \Omega^\bullet(\mathcal{A}), \\ x = \mathsf{d}\Phi, \quad \Phi \in \mathcal{A}. \end{gathered}$$

So that, we obtain the Koszul complex, associated to a given polynomial Φ of $\mathbf{F}[x_1, \ldots, x_n]$,

$$0 \longrightarrow \mathcal{A} \xrightarrow{\wedge \mathsf{d}\Phi} \Omega^1(\mathcal{A}) \xrightarrow{\wedge \mathsf{d}\Phi} \Omega^2(\mathcal{A}) \xrightarrow{\wedge \mathsf{d}\Phi} \cdots$$

We will see that if $\Phi \in \mathbf{F}[x, y]$ is a square free weight homogeneous polynomial (see Paragraph 4.1.3) or if $\Phi \in \mathbf{F}[x, y, z]$ is a weight homogeneous polynomial with an isolated singularity (at the origin) (see Paragraph 3.1.3), then, the partial derivatives of Φ defines a regular sequence, that implies that the Koszul complex associated to Φ is exact (we will prove this result in our particular cases but one can see [20] for general case).

2.3.4 Examples of isolated singularities in dimension three

In this paragraph, we will give many examples of weight homogeneous polynomials φ, in $\mathbf{C}[x, y, z]$, that have an isolated singularity at the origin. To do this, we pick up some singularities from the classification given in [4]. In this book, the singularities are given with their modality, a notion that we do not need in this document. Our purpose is to consider three types of examples, corresponding to the three cases:

$$\varpi(\varphi) < |\varpi|, \qquad \varpi(\varphi) = |\varpi|, \qquad \varpi(\varphi) > |\varpi|,$$

where $|\varpi|$ denotes the sum of the weights of x, y and z: $|\varpi| = \varpi_1 + \varpi_2 + \varpi_3$, ϖ_1 (respectively, ϖ_2, ϖ_3) denoting the weight of x (respectively, y, z). We are interested in such polynomials φ, because, in the next chapter, we will determine the Poisson cohomology of the Poisson variety $(\mathbf{F}^3, \{\cdot\,, \cdot\}_\varphi)$ (See (2.8)), where φ is a weight homogeneous polynomial, with an isolated singularity at the origin. Moreover, we are interested in the sign of $\varpi(\varphi) - |\varpi|$, because this number is exactly the (weighted) degree of the skew-symmetric biderivation $\{\cdot\,, \cdot\}_\varphi$ and we will see in Chapter 3, that the Poisson cohomology spaces depend on its sign. We will denote with a μ the Milnor number of each considered singularity.

Simple singularities

Here are some examples of weight homogeneous isolated singularities in $\mathbf{C}^3$ namely, the so-called *simple singularities.* Under the Mc Kay correspondence, these singularities are related to the simple algebras of type A, D, E (See [4] and [58]). In fact, these singularities are those whose modality is equal to zero.

1. Type $\mathbf{A_k}$, $\mathbf{k \geq 1}$

$$\boxed{\varphi_{A_k} := x^2 + y^2 + z^{k+1}}$$

This polynomial is a weight homogeneous polynomial of $\mathcal{A} = \mathbf{C}[x, y, z]$, with the weights of the variables:

$$\begin{cases} \varpi_1 = \varpi_2 = (k+1)/2, \\ \varpi_3 = 1, \end{cases} \quad \text{if } k \text{ is odd,}$$

$$\begin{cases} \varpi_1 = \varpi_2 = k+1, \\ \varpi_3 = 2, \end{cases} \quad \text{if } k \text{ is even.}$$

We then have $\varpi(\varphi_{A_k}) = k+1$ and $|\varpi| = k+2$, if k is odd; while $\varpi(\varphi_{A_k}) = 2(k+1)$ and $|\varpi| = 2k+4$, if k is even. This polynomial φ_{A_k} has an isolated singularity at the origin and the $\mathbf{C}$-vector space associated to φ_{A_k},

$$\mathcal{A}_{sing}(A_k) = \frac{\mathbf{C}[x, y, z]}{\langle x, y, z^k \rangle}$$

is of finite dimension μ_{A_k}. A $\mathbf{C}$-basis of $\mathcal{A}_{sing}(A_k)$ is indeed given by the family: $1, z, z^2, \cdots, z^{k-1}$, so that $\mu_{A_k} = k$.

2. Type $\mathbf{D_k}$, $\mathbf{k \geq 4}$

$$\boxed{\varphi_{D_k} := x^2 + y^2 z + z^{k-1}}$$

This polynomial is a weight homogeneous one, with the weights:

$$\varpi_1 = k-1, \quad \varpi_2 = k-2, \quad \varpi_3 = 2,$$

so that $\varpi(\varphi_{D_k}) = 2(k-1)$ and $|\varpi| = 2k-1$. The associated $\mathbf{C}$-vector space

$$\mathcal{A}_{sing}(D_k) = \frac{\mathbf{C}[x, y, z]}{\langle x, yz, y^2 + (k-1)z^{k-2} \rangle}$$

admits, as a $\mathbf{C}$-basis, the family: $1, y, z, z^2, \cdots, z^{k-2}$, that leads to $\mu_{D_k} = k$.

3. Type $\mathbf{E_6}$

$$\boxed{\varphi_{E_6} := x^2 + y^3 + z^4}$$

The polynomial φ_{E_6} is a weight homogeneous polynomial, associated to the weights of the variables:

$$\varpi_1 = 6, \quad \varpi_2 = 4, \quad \varpi_3 = 3,$$

that gives $\varpi(\varphi_{E_6}) = 12$, $|\varpi| = 13$. In this case, the singular algebra is

$$\mathcal{A}_{sing}(E_6) = \frac{\mathbf{C}[x,y,z]}{\langle x, y^2, z^3 \rangle}$$

and a $\mathbf{C}$-basis of $\mathcal{A}_{sing}$ is given by: $1, y, z, z^2, yz, yz^2$, so that $\mu_{E_6} = 6$.

4. Type E₇

$$\boxed{\varphi_{E_7} := x^2 + y^3 + yz^3}$$

The weight homogeneity of φ_{E_7} is associated to the weights:

$$\varpi_1 = 9, \quad \varpi_2 = 6, \quad \varpi_3 = 4,$$

so that $\varpi(\varphi_{E_7}) = 18$, $|\varpi| = 19$, and the singular algebra,

$$\mathcal{A}_{sing}(E_7) = \frac{\mathbf{C}[x,y,z]}{\langle x, 3y^2 + z^3, yz^2 \rangle},$$

admits, as a $\mathbf{C}$-basis, the family: $1, y, y^2, z, z^2, yz, y^2z$, then $\mu_{E_7} = 7$.

5. Type E₈

$$\boxed{\varphi_{E_8} := x^2 + y^3 + z^5}$$

This polynomial is also a weight homogeneous one, with the weights of the three variables:

$$\varpi_1 = 15, \quad \varpi_2 = 10, \quad \varpi_3 = 6,$$

and the sum of these weights $|\varpi| = 31$. The weighted degree of φ_{E_8} is $\varpi(\varphi_{E_8}) = 30$, while the associated singular algebra is given by:

$$\mathcal{A}_{sing}(E_8) = \frac{\mathbf{C}[x,y,z]}{\langle x, y^2, z^4 \rangle}.$$

It is a finite dimensional $\mathbf{C}$-vector space and the family: $1, y, z, z^2, z^3, yz, yz^2, yz^3$ gives a $\mathbf{C}$-basis of it, so that the dimension of $\mathcal{A}_{sing}$ is $\mu_{E_8} = 8$.

Remark 2.32. For each of these singularities φ, we have

$$\varpi(\varphi) < |\varpi|.$$

When we will determine the Poisson cohomology spaces associated to weight homogeneous polynomials with an isolated singularity, we will see that these kind of inequality will play an important role.

Some unimodal singularities

Now, we will consider two examples of singularities of modality equal to one.

1. Parabolic homogeneous case

$$\boxed{\varphi_{ph} := x^3 + y^3 + z^3 + a\,xyz,\ a^3 + 27 \neq 0}$$

This particular φ_{ph} is a homogeneous polynomial of $\mathbf{C}[x, y, z]$, the weights of the three variables are equal to one:

$$\varpi_1 = \varpi_2 = \varpi_3 = 1$$

and the degree of φ_{ph} is $\varpi(\varphi_{ph}) = 3$, so that $\varpi(\varphi_{ph}) = |\varpi|$. The singular algebra

$$\mathcal{A}_{sing}(\varphi_{ph}) = \frac{\mathbf{C}[x, y, z]}{\left\langle 3x^2 + ayz,\, 3y^2 + axz,\, 3z^2 + axy \right\rangle}$$

is a finite dimensional $\mathbf{C}$-vector space and admits, as a $\mathbf{C}$-basis, the family: $1, x, y, z, xy, xz, yz, xyz$. As a consequence, the Milnor number of this singularity is equal to $\mu_{ph} = 8$.

2. Parabolic weight homogeneous case

$$\boxed{\varphi_{pwh} := x^2 + y^4 + z^4 + b\,y^2z^2,\ b^2 \neq 4}$$

With the weights

$$\varpi_1 = 2, \quad \varpi_2 = \varpi_3 = 1,$$

this polynomial is a weight homogeneous one, of degree $\varpi(\varphi_{pwh}) = 4 = |\varpi|$. The quotient algebra

$$\mathcal{A}_{sing}(\varphi_{pwh}) = \frac{\mathbf{C}[x, y, z]}{\left\langle x,\, y(2y^2 + bz^2),\, z(2z^2 + by^2) \right\rangle}$$

is a finite dimensional $\mathbf{C}$-vector space, with $\mathbf{C}$-basis:

$$1, y, z, y^2, z^2, yz, yz^2, zy^2, y^2z^2.$$

Its dimension is then equal to $\mu_{pwh} = 9$.

Remark 2.33. The two last examples considered were weight homogeneous polynomials φ for which we have the following inequality:

$$\varpi(\varphi) = |\varpi|.$$

We will see in Section 3.2 that the polynomials satisfying this property are those which are associated to Poisson structures of (weighted) degree equal to zero and with a Poisson cohomology that has some particular properties.

3. Among the exceptional families, type $\mathbf{E_{12}}$

$$\boxed{\varphi_{E_{12}} := x^2 + y^3 + z^7}$$

Considering the variables equipped with the weights:

$$\varpi_1 = 21, \quad \varpi_2 = 14, \quad \varpi_3 = 6,$$

the polynomial $\varphi_{E_{12}}$ is a weight homogeneous one, of degree $\varpi(\varphi_{E_{12}}) = 42$, while the sum of the weights of the three variables is equal to $|\varpi| = 41$. The quotient algebra, associated with $\varphi_{E_{12}}$

$$\mathcal{A}_{sing}(E_{12}) = \frac{\mathbf{C}[x, y, z]}{\langle x,\, y^2,\, z^6 \rangle}$$

is a finite dimensional $\mathbf{C}$-vector space, which admits the family:

$$1, y, z, z^2, z^3, z^4, z^5, yz, yz^2, yz^3, yz^4, yz^5$$

as a $\mathbf{C}$-basis. The Milnor number of this singularity is then equal to $\mu_{E_{12}} = 12$.

Some bimodal singularities

We here study some bimodal examples.

1. Among the families of corank 2, type $\mathbf{W_{1,0}}$

$$\boxed{\varphi_{W_{1,0}} := x^2 + y^4 + d\, y^2 z^3 + z^6,\ d^2 \neq 4}$$

We consider the weights of the variables

$$\varpi_1 = 6, \quad \varpi_2 = 3, \quad \varpi_3 = 2,$$

so that $\varphi_{W_{1,0}}$ is a weight homogeneous polynomial of degree $\varpi(\varphi_{W_{1,0}}) = 12$ and $|\varpi| = 11$. The associated quotient algebra

$$\mathcal{A}_{sing}(W_{1,0}) = \frac{\mathbf{C}[x, y, z]}{\left\langle x,\, y(2y^2 + d\, z^3),\, z^2(d\, y^2 + 2z^3) \right\rangle}$$

is a finite dimensional $\mathbf{C}$-vector space, with a $\mathbf{C}$-basis:

$$1, y, z, y^2, z^2, yz, z^3, z^4, yz^2, yz^3, yz^4, y^2z, y^2z^2, y^2z^3, y^2z^4,$$

and the Milnor number is then equal to $\mu_{W_{1,0}} = 15$.

2. Among the exceptional families, type $\mathbf{W_{17}}$

$$\boxed{\varphi_{W_{17}} := x^2 + y^4 + yz^5}$$

This polynomial is a weight homogeneous one, with the weights:

$$\varpi_1 = 10, \quad \varpi_2 = 5, \quad \varpi_3 = 3,$$

so that $|\varpi| = 18$ and $\varpi(\varphi_{W_{17}}) = 20$. The associated quotient algebra

$$\mathcal{A}_{sing}(W_{17}) = \frac{\mathbf{C}[x, y, z]}{\left\langle x,\, 4y^3 + z^5,\, yz^4 \right\rangle}$$

is, viewed as a **C**-vector space, of finite dimension and admits the family

$$1, y, y^2, y^3, z, z^2, z^3, z^4, yz, yz^2, yz^3, y^2z, y^2z^2, y^2z^3, y^3z, y^3z^2, y^3z^3,$$

as a **C**-basis, so that, the Milnor number is, in this case, equal to $\mu_{W_{17}} = 17$.

Remark 2.34. The two last examples of singularities φ (of type E_{12} and $W_{1,0}$) satisfy the inequality:

$$\varpi(\varphi) > |\varpi|.$$

3

Poisson cohomology and homology of the affine space $\mathbf{F}^3$

In this chapter, we consider the affine space of dimension three $\mathbf{F}^3$ and its algebra of regular functions $\mathcal{A} = \mathbf{F}[x, y, z]$. To each polynomial $\varphi \in \mathcal{A}$, one associates a Poisson structure on $\mathbf{F}^3$, denoted by $\{\cdot\,,\cdot\}_\varphi$. Under the hypotheses of weight homogeneity and isolated singularity at the origin for φ, we determine the Poisson cohomology and homology of the Poisson variety $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$ and see how the singularity appears in the Poisson (co)homology spaces. To do this, we first begin by study the skew-symmetric multi-derivations and the Kähler differentials in dimension three.

3.1 Multi-Derivations and Kähler differentials in dimension three

In this paragraph, we consider the skew-symmetric multi-derivations and the Kähler differentials of the polynomial algebra $\mathcal{A} = \mathbf{F}[x, y, z]$. In order to simplify the notations, we will give some identifications that will be useful when we will consider Poisson cohomology of this algebra.

3.1.1 Multi-derivations

We consider the polynomial algebra $\mathcal{A} = \mathbf{F}[x, y, z]$ and the skew-symmetric multi-derivations of $\mathcal{A}$. By convention, $\mathfrak{X}^0(\mathcal{A}) = \mathcal{A}$ and, according to Remark 2.14, we have $\mathfrak{X}^k(\mathcal{A}) \simeq \{0\}$, as soon as $k \geq 4$. Let $\mathcal{V} \in \mathfrak{X}^1(\mathcal{A})$ be a derivation of the algebra $\mathcal{A}$. Then, according to Proposition 2.15, it is totally given by the three polynomials:

$$F_1 := \mathcal{V}[x], \qquad F_2 := \mathcal{V}[y], \qquad F_3 := \mathcal{V}[z], \qquad (F_1, F_2, F_3) \in \mathcal{A}^3,$$

so that:

$$\mathcal{V} = F_1 \frac{\partial}{\partial x} + F_2 \frac{\partial}{\partial y} + F_3 \frac{\partial}{\partial z}.$$

Thus, we have the correspondence $\mathfrak{X}^1(\mathcal{A}) \simeq \mathcal{A}^3$:

$$\begin{array}{ccc} \mathfrak{X}^1(\mathcal{A}) & \longleftrightarrow & \mathcal{A}^3 \\ \mathcal{V} & \longrightarrow & (\mathcal{V}[x], \mathcal{V}[y], \mathcal{V}[z]) \\ F_1 \frac{\partial}{\partial x} + F_2 \frac{\partial}{\partial y} + F_3 \frac{\partial}{\partial z} & \longleftarrow & (F_1, F_2, F_3) \end{array}$$

We do the same reasoning for the skew-symmetric biderivations of $\mathcal{A}$. Let $\mathcal{W} \in \mathfrak{X}^2(\mathcal{A})$ and

$$G_1 := \mathcal{W}[y,z], \qquad G_2 := \mathcal{W}[z,x], \qquad G_3 := \mathcal{W}[x,y], \qquad (G_1, G_2, G_3) \in \mathcal{A}^3,$$

then, necessarily we have:

$$\mathcal{W} = G_1 \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + G_2 \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + G_3 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

The space $\mathfrak{X}^2(\mathcal{A})$ can so be identified with $\mathcal{A}^3$ as follows:

$$\begin{array}{ccc} \mathfrak{X}^2(\mathcal{A}) & \longleftrightarrow & \mathcal{A}^3 \\ \mathcal{W} & \longrightarrow & (\mathcal{W}[y,z], \mathcal{W}[z,x], \mathcal{W}[x,y]) \\ G_1 \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + G_2 \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + G_3 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} & \longleftarrow & (G_1, G_2, G_3) \end{array}$$

Finally, let $\mathcal{Z} \in \mathfrak{X}^3(\mathcal{A})$ be a skew-symmetric 3-derivation of $\mathcal{A}$ and let

$$H = \mathcal{Z}[x,y,z] \in \mathcal{A},$$

then, we have:

$$\mathcal{Z} = H \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z},$$

and we can write the following correspondence between $\mathfrak{X}^3(\mathcal{A})$ and $\mathcal{A}$:

$$\begin{array}{ccc} \mathfrak{X}^3(\mathcal{A}) & \longleftrightarrow & \mathcal{A} \\ \mathcal{Z} & \longrightarrow & \mathcal{Z}[x,y,z] \\ H \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} & \longleftarrow & H \end{array}$$

Vector Formulas

In this part, we use the notations and the formulas of the vector calculus in $\mathbf{R}^3$, adapted to $\mathcal{A}^3$. Recall that we have obtained in the last paragraph the following isomorphisms:

$$\mathfrak{X}^0(\mathcal{A}) \simeq \mathfrak{X}^3(\mathcal{A}) \simeq \mathcal{A}; \quad \mathfrak{X}^1(\mathcal{A}) \simeq \mathfrak{X}^2(\mathcal{A}) \simeq \mathcal{A}^3; \quad \mathfrak{X}^k(\mathcal{A}) \simeq \{0\}, \text{ for } k \geq 4.$$

The elements of $\mathcal{A}^3$ are viewed as vector-valued functions on $\mathcal{A}$, so we denote them with an arrow, like $\vec{F} \in \mathcal{A}^3$. Sometimes, it will be important to distinguish

$\mathcal{A}^3 \simeq \mathfrak{X}^1(\mathcal{A})$ from $\mathcal{A}^3 \simeq \mathfrak{X}^2(\mathcal{A})$; then we will rather write $\vec{F} \in \mathfrak{X}^1(\mathcal{A})$ or $\vec{F} \in \mathfrak{X}^2(\mathcal{A})$.

As done in Section 2.1.3, in $\mathcal{A}^3$, let $\cdot$, $\times$ denote respectively the usual inner and cross products, while $\vec{\nabla}$, $\vec{\nabla}\times$, Div denote respectively the gradient, the curl and the divergence operators. More precisely, let F be an element of $\mathcal{A}$ and $\vec{F}$, $\vec{G}$, be two elements of $\mathcal{A}^3$, with: $\vec{F} = (F_1, F_2, F_3)$, $\vec{G} = (G_1, G_2, G_3)$. The products $\cdot$ and $\times$ are defined by:

$$\vec{F} \cdot \vec{G} := F_1\, G_1 + F_2\, G_2 + F_3\, G_3 \in \mathcal{A},$$

$$\vec{F} \times \vec{G} := (F_2\, G_3 - F_3\, G_2,\ F_3\, G_1 - F_1\, G_3,\ F_1\, G_2 - F_2\, G_1) \in \mathcal{A}^3.$$

The operator $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ leads to the gradient of F

$$\vec{\nabla} F := \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) \in \mathcal{A}^3,$$

and to the curl operator of $\vec{F}$

$$\vec{\nabla} \times \vec{F} := \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \in \mathcal{A}^3.$$

Finally, the divergence of $\vec{F}$ is given by:

$$\mathrm{Div}(\vec{F}) := \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \in \mathcal{A}.$$

In the further chapters, we will often use, for $\vec{F}, \vec{G} \in \mathcal{A}^3$ and $F, G, H \in \mathcal{A}$, the following formulas, well-known from vector calculus in $\mathbf{R}^3$:

$$\vec{\nabla} \times (F\vec{G}) = \vec{\nabla} F \times \vec{G} + F(\vec{\nabla} \times \vec{G}), \tag{3.1}$$

$$\mathrm{Div}(F\vec{G}) = \vec{\nabla} F \cdot \vec{G} + F\, \mathrm{Div}(\vec{G}), \tag{3.2}$$

$$\mathrm{Div}(\vec{F} \times \vec{G}) = (\vec{\nabla} \times \vec{F}) \cdot \vec{G} - \vec{F} \cdot (\vec{\nabla} \times \vec{G}), \tag{3.3}$$

$$\vec{\nabla} F \cdot \left(\vec{\nabla} G \times \vec{\nabla} H\right) = \vec{\nabla} G \cdot \left(\vec{\nabla} H \times \vec{\nabla} F\right) = \vec{\nabla} H \cdot \left(\vec{\nabla} F \times \vec{\nabla} G\right). \tag{3.4}$$

Weight homogeneity in dimension three

We give, in this part, some useful facts that appear in a weight homogeneous context, in dimension three. Suppose that we have fixed weights for the variables x, y, z: $\varpi_1 = \varpi(x)$, $\varpi_2 = \varpi(y)$, $\varpi_3 = \varpi(z)$. The associated Euler derivation $\vec{e}_\varpi$ (see (2.35)) is then identified, with the isomorphisms given at the beginning of this paragraph, to the element (also denoted by $\vec{e}_\varpi$),

$$\vec{e}_\varpi = (\varpi_1\, x, \varpi_2\, y, \varpi_3\, z) \in \mathcal{A}^3.$$

We denote by $|\varpi|$ the sum of the weights $\varpi_1+\varpi_2+\varpi_3$, so that

$$|\varpi| = \mathrm{Div}(\vec{e}_\varpi).$$

Euler's formula (2.36), for a weight homogeneous $F \in \mathcal{A}$, can now be written as

$$\vec{\nabla} F \cdot \vec{e}_\varpi = \varpi(F)\, F, \tag{3.5}$$

and yields, using (3.2):

$$\mathrm{Div}(F\vec{e}_\varpi) = (\varpi(F) + |\varpi|)\, F. \tag{3.6}$$

If $\varpi_1=\varpi_2=\varpi_3=1$, i.e., in the homogeneous context, the element $\vec{e}_\varpi$ is rather denoted by $\vec{e}$.

Fixing weights $\varpi_1, \varpi_2, \varpi_3 \in \mathbf{N}^*$, using the notations of paragraph 2.3.1, the correspondences in the previous paragraph, and the remark 2.31, we have the following isomorphisms:

$$\begin{aligned}
&\mathfrak{X}^0(\mathcal{A})_i \simeq \mathcal{A}_i,\\
&\mathfrak{X}^1(\mathcal{A})_i \simeq \mathcal{A}_{i+\varpi_1} \times \mathcal{A}_{i+\varpi_2} \times \mathcal{A}_{i+\varpi_3},\\
&\mathfrak{X}^2(\mathcal{A})_i \simeq \mathcal{A}_{i+\varpi_2+\varpi_3} \times \mathcal{A}_{i+\varpi_1+\varpi_3} \times \mathcal{A}_{i+\varpi_1+\varpi_2},\\
&\mathfrak{X}^3(\mathcal{A})_i \simeq \mathcal{A}_{i+\varpi_1+\varpi_2+\varpi_3}.
\end{aligned} \tag{3.7}$$

Notice that even if $\mathfrak{X}^1(\mathcal{A}) \simeq \mathfrak{X}^2(\mathcal{A}) \simeq \mathcal{A}^3$ and $\mathfrak{X}^0(\mathcal{A}) \simeq \mathfrak{X}^3(\mathcal{A}) \simeq \mathcal{A}$, these isomorphisms do not respect the weight decompositions (3.7).

3.1.2 Kähler differentials

As for the skew-symmetric multi-derivations, we can consider the particular case of the Kähler differentials of $\mathcal{A} = \mathbf{F}[x,y,z]$ and obtain some identifications.

We have $\Omega^0(\mathcal{A}) = \mathcal{A}$ and, for all $k \geq 4$, $\Omega^k(\mathcal{A}) = \{0\}$. According to the definition of the Kähler differentials, we have the following correspondences:

$$\begin{array}{ccc}
\Omega^1(\mathcal{A}) & \longleftrightarrow & \mathcal{A}^3\\
G_1\,\mathsf{d}x + G_2\,\mathsf{d}y + G_3\,\mathsf{d}z & \longleftrightarrow & (G_1, G_2, G_3)
\end{array}$$

$$\begin{array}{ccc}
\Omega^2(\mathcal{A}) & \longleftrightarrow & \mathcal{A}^3\\
F_1\,\mathsf{d}y\wedge\mathsf{d}z + F_2\,\mathsf{d}z\wedge\mathsf{d}x + F_3\,\mathsf{d}x\wedge\mathsf{d}y & \longleftrightarrow & (F_1, F_2, F_3)
\end{array}$$

$$\begin{array}{ccc}
\Omega^3(\mathcal{A}) & \longleftrightarrow & \mathcal{A}\\
K\,\mathsf{d}x\wedge\mathsf{d}y\wedge\mathsf{d}z & \longleftrightarrow & K
\end{array}$$

So that, we have the isomorphisms (See Paragraph 3.1.1 for the skew-symmetric multi-derivations of $\mathcal{A} = \mathbf{F}[x, y, z]$):

$$\begin{aligned} \Omega^0(\mathcal{A}) &\simeq \mathcal{A} \simeq \mathfrak{X}^3(\mathcal{A}), \\ \Omega^1(\mathcal{A}) &\simeq \mathcal{A}^3 \simeq \mathfrak{X}^2(\mathcal{A}), \\ \Omega^2(\mathcal{A}) &\simeq \mathcal{A}^3 \simeq \mathfrak{X}^1(\mathcal{A}), \\ \Omega^3(\mathcal{A}) &\simeq \mathcal{A} \simeq \mathfrak{X}^0(\mathcal{A}). \end{aligned} \tag{3.8}$$

Because of these identifications, a Kähler 1 or 2-differential of $\mathcal{A}$, viewed as an element of $\mathcal{A}^3$, will be denoted with an arrow, as for the skew-symmetric 2 or 1-derivations of $\mathcal{A}$. We will also use the same notations ($\vec{\nabla}$, $\times$, Div, ...) as those introduced in Paragraph 3.1.1 for the skew-symmetric multi-derivations, adapted to the Kähler differentials.

More precisely, the isomorphisms between $\mathfrak{X}^k(\mathcal{A})$ and $\Omega^{3-k}(\mathcal{A})$ (for $0 \leq k \leq 3$) are given by the star operator $\star$ (see Paragraph 2.2.2) and are explicitly written as follows:

$$\begin{array}{ccc} \Omega^0(\mathcal{A}) & \stackrel{*}{\longleftrightarrow} & \mathfrak{X}^3(\mathcal{A}) \\ H & \longleftrightarrow & H\,\dfrac{\partial}{\partial x} \wedge \dfrac{\partial}{\partial y} \wedge \dfrac{\partial}{\partial z} \end{array}$$

$$\begin{array}{ccc} \Omega^1(\mathcal{A}) & \stackrel{*}{\longleftrightarrow} & \mathfrak{X}^2(\mathcal{A}) \\ G_1\,\mathsf{d}x + G_2\,\mathsf{d}y + G_3\,\mathsf{d}z & \longleftrightarrow & G_1\,\dfrac{\partial}{\partial y} \wedge \dfrac{\partial}{\partial z} + G_2\,\dfrac{\partial}{\partial z} \wedge \dfrac{\partial}{\partial x} \\ & & +G_3\,\dfrac{\partial}{\partial x} \wedge \dfrac{\partial}{\partial y} \end{array}$$

$$\begin{array}{ccc} \Omega^2(\mathcal{A}) & \stackrel{*}{\longleftrightarrow} & \mathfrak{X}^1(\mathcal{A}) \\ F_1\,\mathsf{d}y \wedge \mathsf{d}z + F_2\,\mathsf{d}z \wedge \mathsf{d}x + F_3\,\mathsf{d}x \wedge \mathsf{d}y & \longleftrightarrow & F_1\dfrac{\partial}{\partial x} + F_2\,\dfrac{\partial}{\partial y} + F_3\,\dfrac{\partial}{\partial z} \end{array}$$

$$\begin{array}{ccc} \Omega^3(\mathcal{A}) & \stackrel{*}{\longleftrightarrow} & \mathfrak{X}^0(\mathcal{A}) \\ K\,\mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z & \longleftrightarrow & K \end{array}$$

According to the previous isomorphisms, we can write the de Rham complex, in dimension three, in terms of elements of $\mathcal{A}$ and $\mathcal{A}^3$.

If $F \in \Omega^0(\mathcal{A}) = \mathcal{A}$, then $\mathsf{d}F \in \Omega^1(\mathcal{A})$ can be identified, according to the previous isomorphisms, with the element $\vec{\nabla} F = \left(\dfrac{\partial F}{\partial x}, \dfrac{\partial F}{\partial y}, \dfrac{\partial F}{\partial z}\right) \in \mathcal{A}^3$.

For $\alpha = F_1\,\mathsf{d}x + F_2\,\mathsf{d}y + F_3\,\mathsf{d}z \in \Omega^1(\mathcal{A})$, with $\vec{F} := (F_1, F_2, F_3) \in \mathcal{A}^3$, the de Rham differential gives

$$\mathsf{d}\alpha = \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\, \mathsf{d}y \wedge \mathsf{d}z + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\, \mathsf{d}z \wedge \mathsf{d}x + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\, \mathsf{d}x \wedge \mathsf{d}y \ \in \Omega^2(\mathcal{A}),$$

that one can see as the element $\vec{\nabla} \times \vec{F} \in \mathcal{A}^3$ (see the definition of the curl operator in the paragraph 3.1.1).

Now, let $\beta = G_1\, \mathsf{d}y \wedge \mathsf{d}z + G_2\, \mathsf{d}z \wedge \mathsf{d}x + G_3\, \mathsf{d}x \wedge \mathsf{d}y \in \Omega^2(\mathcal{A})$ be a Kähler 2-differential of $\mathcal{A}$, where $\vec{G} := (G_1, G_2, G_3) \in \mathcal{A}^3$. Then,

$$\mathsf{d}\beta = \Big(\frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z}\Big)\, \mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z \ \in \Omega^3(\mathcal{A}),$$

that is identified with the element $\mathrm{Div}(\vec{G}) = \Big(\frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z}\Big) \in \mathcal{A}$.

Of course, if $\gamma \in \Omega^3(\mathcal{A})$ is a Kähler 3-differential, then $\mathsf{d}\gamma = 0$. So we obtain the de Rham complex of $\mathcal{A} = \mathbf{F}[x, y, z]$, expressed in terms of elements of $\mathcal{A}$ and $\mathcal{A}^3$ (see paragraph 3.1.1 for the notations used in $\mathcal{A}^3$):

$$\begin{array}{ccccccccc}
\mathbf{F} \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{A}^3 & \longrightarrow & \mathcal{A}^3 & \longrightarrow & \mathcal{A} & \longrightarrow 0 \\
& F & \longmapsto & \vec{\nabla} F & & & & & \\
& & & \vec{F} & \longmapsto & \vec{\nabla} \times \vec{F} & & & \\
& & & & & \vec{G} & \longmapsto & \mathrm{Div}(\vec{G}) &
\end{array} \tag{3.9}$$

Proposition 3.1. *The algebraic de Rham complex (3.9) of the polynomial algebra* $\mathcal{A} = \mathbf{F}[x, y, z]$ *is an exact one.*

Proof. In this proof, we will work with the writing of the de Rham complex of $\mathcal{A}$, given in (3.9). The classical argument of exactness of the de Rham complex of $C^\infty(\mathbf{R}^n)$ will be adapted to the algebraic case.

Let $\vec{F} = (F_1, F_2, F_3) \in \mathcal{A}^3$ be composed of three homogeneous polynomials of degree $r \in \mathbf{N}$, with $\mathrm{Div}(\vec{F}) = 0$. We recall that $\vec{e}$ denotes the Euler derivation (2.35) in the homogeneous context (when $\varpi_1 = \varpi_2 = \varpi_3 = 1$). Then, $\mathrm{Div}(\vec{F}) = 0$ implies that the first component of $\vec{\nabla} \times (\vec{F} \times \vec{e})$ is equal to

$$\begin{aligned}
\Big(\vec{\nabla} \times (\vec{F} \times \vec{e})\Big)_1 &= 2F_1 + y\frac{\partial F_1}{\partial y} + z\frac{\partial F_1}{\partial z} - x\big(\frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\big) \\
&= 2F_1 + y\frac{\partial F_1}{\partial y} + z\frac{\partial F_1}{\partial z} + x\frac{\partial F_1}{\partial x} \\
&= (2 + r)\, F_1,
\end{aligned}$$

in view of Euler's Formula (2.36) in the homogeneous context. Analogous equalities hold for the second and the third components of $\vec{\nabla} \times (\vec{F} \times \vec{e})$, so that we obtain that $\vec{F} = \vec{\nabla} \times \vec{G}$, where $\vec{G} = \frac{1}{r+2}(\vec{F} \times \vec{e}) \in \mathcal{A}^3$.

Similarly, $\vec{\nabla} \times \vec{F} = \vec{0}$ leads to

$$\begin{aligned}
\left(\vec{\nabla}(\vec{F}\cdot\vec{e})\right)_1 &= \frac{\partial}{\partial x}\left(x\,F_1 + y\,F_2 + z\,F_3\right) \\
&= F_1 + x\,\frac{\partial F_1}{\partial x} + y\,\frac{\partial F_2}{\partial x} + z\,\frac{\partial F_3}{\partial x} \\
&= F_1 + x\,\frac{\partial F_1}{\partial x} + y\,\frac{\partial F_1}{\partial y} + z\,\frac{\partial F_1}{\partial z} \\
&= (r+1)\,F_1,
\end{aligned}$$

according to Euler's Formula (2.36) one more time. As we have analogous computations for the other components of $\vec{\nabla}(\vec{F}\cdot\vec{e})$, this computation leads to $\vec{F}=\vec{\nabla}H$, with $H = \frac{1}{r+1}(\vec{F}\cdot\vec{e}) \in \mathcal{A}$. We have so obtained that the algebraic de Rham complex of $\mathcal{A}$ is exact. $\square$

3.1.3 The Koszul complex in dimension three

Let us now consider the affine space of dimension three, $\mathbf{F}^3$ and let us denote by $\mathcal{A}$ its algebra of regular functions $\mathcal{A} := \mathbf{F}[x,y,z]$. In this paragraph, $\varphi \in \mathcal{A}$ will denote a weight homogeneous polynomial with an isolated singularity.

With the Cohen-Macaulay theorem (see [56] and [57] for proofs), we will see that, in this case (where φ is weight homogeneous with an isolated singularity), the Koszul complex associated to φ (see Paragraph 2.3.3) is also exact. For this purpose, we will see that the sequence of partial derivatives of φ: $\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}$ is a regular sequence of $\mathcal{A}$. In order to explain that, we first have to write down the definition of a homogeneous system of parameters of an algebra.

Definition 3.2. *Let $\mathcal{A}$ be an associative and commutative graded $\mathbf{F}$-algebra. A system of homogeneous elements $F_1,\ldots,F_d$ in $\mathcal{A}$, where d is the Krull dimension of $\mathcal{A}$, is called a homogeneous system of parameters of $\mathcal{A}$ (h.s.o.p.) if $\mathcal{A}/\langle F_1,\ldots,F_d\rangle$ is a finite dimensional $\mathbf{F}$-vector space.*

For example, if we consider the $\mathbf{F}$-algebra $\mathcal{A}=\mathbf{F}[x,y,z]$, which is graded by the weighted degree associated to φ, we have a natural h.s.o.p. given by the system x,y,z. Moreover, we have seen in Paragraph 2.3.2 that a weight homogeneous element $\varphi\in\mathcal{A}$ has an isolated singularity if and only if the three partial derivatives $\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}$ give a h.s.o.p. of $\mathcal{A}$.

In order to understand the following theorem, that we will need, we still have to give the definition of a regular sequence.

Definition 3.3. *A sequence $a_1,\ldots,a_n$ in a commutative associative algebra $\mathcal{A}$ is said to be a $\mathcal{A}$-regular sequence if $\langle a_1,\ldots,a_n\rangle \neq \mathcal{A}$ and a_i is not a zero divisor of $\mathcal{A}/\langle a_1,\ldots,a_{i-1}\rangle$ for $i=1,2,\ldots,n$.*

For example, it is clear that the sequence x,y,z is a regular sequence in $\mathbf{F}[x,y,z]$. But, what about $\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}$, when φ is weight homogeneous with an isolated singularity?

Theorem 3.4 (Cohen-Macaulay). *Let $\mathcal{A}$ be a Noetherian graded $\mathbf{F}$-algebra. If $\mathcal{A}$ has a h.s.o.p. which is a regular sequence, then any h.s.o.p. in $\mathcal{A}$ is a regular sequence.*

Thus, when $\varphi \in \mathcal{A} = \mathbf{F}[x,y,z]$ is a weight homogeneous polynomial with an isolated singularity, then $\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}$ is a regular sequence. This is the key fact that permits us to prove the next proposition, saying that the Koszul complex associated to φ is an exact one.

According to the identifications of the paragraph 3.1.2, we can rewrite the Koszul complex associated to $\varphi \in \mathcal{A} = \mathbf{F}[x,y,z]$ as follows:

$$\{0\} \longrightarrow \mathcal{A} \xrightarrow{\vec{\nabla}\varphi} \mathcal{A}^3 \xrightarrow{\times\vec{\nabla}\varphi} \mathcal{A}^3 \xrightarrow{\cdot\vec{\nabla}\varphi} \mathcal{A} \tag{3.10}$$

For example, if $\vec{F} \in \Omega^1(\mathcal{A}) \simeq \mathcal{A}^3$, $\vec{F} = F_1\mathsf{d}x + F_2\mathsf{d}y + F_3\mathsf{d}z$, then

$$\begin{aligned}\vec{F} \wedge \mathsf{d}\varphi = &\left(F_2\frac{\partial\varphi}{\partial z} - F_3\frac{\partial\varphi}{\partial y}\right)\mathsf{d}y\wedge\mathsf{d}z + \left(F_3\frac{\partial\varphi}{\partial x} - F_1\frac{\partial\varphi}{\partial z}\right)\mathsf{d}z\wedge\mathsf{d}x\\ &+\left(F_1\frac{\partial\varphi}{\partial y} - F_2\frac{\partial\varphi}{\partial 1}\right)\mathsf{d}x\wedge\mathsf{d}y\end{aligned}$$

which is identified, under the isomorphisms of Paragraph 3.1.2, with the element $\vec{F} \times \vec{\nabla}\varphi \in \mathcal{A}^3$. It is well-known that a Koszul complex associated to a regular sequence (see Paragraph 2.3.3 and [20]) is exact, but, as it is very simple to prove this result in the particular case of the Koszul complex (3.10) and as it gives a good idea of what happens, we will here do it.

Proposition 3.5. *If $\varphi \in \mathcal{A}$ is a weight homogeneous polynomial, with an isolated singularity, then the Koszul complex (3.10) is exact.*

Proof. Let us prove that the Koszul complex, associated to $\varphi \in \mathcal{A}$ is exact, when φ is weight homogeneous with an isolated singularity. If $\vec{F} = (F_1, F_2, F_3) \in \mathcal{A}^3$ satisfies the equation $\vec{F} \times \vec{\nabla}\varphi = \vec{0}$, then we have the three equalities:

$$F_1\frac{\partial\varphi}{\partial y} - F_2\frac{\partial\varphi}{\partial x} = 0, \quad F_2\frac{\partial\varphi}{\partial z} - F_3\frac{\partial\varphi}{\partial y} = 0, \quad F_3\frac{\partial\varphi}{\partial x} - F_1\frac{\partial\varphi}{\partial z} = 0.$$

Let us consider the first one. Since the partial derivatives of φ form a regular sequence, $\frac{\partial\varphi}{\partial y}$ is not a zero divisor in $\mathcal{A}/\langle\frac{\partial\varphi}{\partial x}\rangle$, so there exists $\alpha \in \mathcal{A}$ such that $F_1 = \alpha\frac{\partial\varphi}{\partial x}$ and then $F_2 = \alpha\frac{\partial\varphi}{\partial y}$. The other equations imply that $F_3 = \alpha\frac{\partial\varphi}{\partial z}$, that is to say $\vec{F} = \alpha\vec{\nabla}\varphi$.

For the second part of the exactitude of the Koszul complex, the reasoning is exactly of the same kind. Assume that $\vec{F} = (F_1, F_2, F_3) \in \mathcal{A}^3$ satisfies the equation $\vec{F} \cdot \vec{\nabla}\varphi = 0$, i.e.,

$$F_1\frac{\partial\varphi}{\partial x}+F_2\frac{\partial\varphi}{\partial y}=-F_3\frac{\partial\varphi}{\partial z}.$$

As the sequence $\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial y},\frac{\partial\varphi}{\partial z}$ is regular, $\frac{\partial\varphi}{\partial z}$ is not a zero divisor in $\mathcal{A}/\langle\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial y}\rangle$, so that the above equation leads to the existence of $G,H\in\mathcal{A}$ satisfying: $F_3=G\frac{\partial\varphi}{\partial x}+H\frac{\partial\varphi}{\partial y}$ and

$$\Big(F_1+G\frac{\partial\varphi}{\partial z}\Big)\frac{\partial\varphi}{\partial x}=-\Big(F_2+H\frac{\partial\varphi}{\partial z}\Big)\frac{\partial\varphi}{\partial y}.$$

With the same reasoning than above, we obtain the existence of $K\in\mathcal{A}$ such that: $F_2+H\frac{\partial\varphi}{\partial z}=K\frac{\partial\varphi}{\partial x}$ and then $F_1+G\frac{\partial\varphi}{\partial z}=-K\frac{\partial\varphi}{\partial y}$, so that $\vec{F}=\vec{H}\times\vec{\nabla}\varphi$, where $\vec{H}=\Big(H,-G,K\Big)\in\mathcal{A}^3$. Thus the Koszul complex is exact. □

Associating Proposition 3.1 with the above Proposition 3.5, we obtain the following result, that will play a fundamental role in our computations of Poisson cohomology, associated to a polynomial.

Proposition 3.6. *For any $\varphi\in\mathcal{A}$ the following diagram*

$$\begin{array}{cccccccccc}
 & & & \mathbf{F} & & \mathcal{A} & & \mathcal{A}^3 & & \\
 & & & \downarrow & & \downarrow{\scriptstyle\vec{\nabla}} & & \downarrow{\scriptstyle\vec{\nabla}\times} & & \\
 & & 0 & \longrightarrow & \mathcal{A} & \xrightarrow{\vec{\nabla}\varphi} & \mathcal{A}^3 & \xrightarrow{\times\vec{\nabla}\varphi} & \mathcal{A}^3 & \xrightarrow{\cdot\vec{\nabla}\varphi} & \mathcal{A} \\
 & & & & \downarrow{\scriptstyle\vec{\nabla}} & & \downarrow{\scriptstyle\vec{\nabla}\times} & & \downarrow{\scriptstyle\mathrm{Div}} & & \\
 & & \mathcal{A} & \xrightarrow{\vec{\nabla}\varphi} & \mathcal{A}^3 & \xrightarrow{\times\vec{\nabla}\varphi} & \mathcal{A}^3 & \xrightarrow{\cdot\vec{\nabla}\varphi} & \mathcal{A} & & \\
 & & \downarrow{\scriptstyle\vec{\nabla}} & & \downarrow{\scriptstyle\vec{\nabla}\times} & & \downarrow{\scriptstyle\mathrm{Div}} & & & & \\
\mathcal{A} & \xrightarrow{\vec{\nabla}\varphi} & \mathcal{A}^3 & \xrightarrow{\times\vec{\nabla}\varphi} & \mathcal{A}^3 & \xrightarrow{\cdot\vec{\nabla}\varphi} & \mathcal{A} & & & &
\end{array}$$

is commutative and has exact columns. If φ is weight homogeneous with an isolated singularity, then the rows of this diagram are also exact.

Remark 3.7. If $\varphi\in\mathcal{A}$ is weight homogeneous, then, as maps from $\mathfrak{X}^k(\mathcal{A})$ to $\mathfrak{X}^{k-1}(\mathcal{A})$, each of the vertical arrows is weight homogeneous of degree zero, while each of the horizontal arrows is weight homogeneous of degree $\varpi(\varphi)$, the (weighted) degree of φ, leading to:

$$\begin{array}{ccccccc}
 & & \mathfrak{X}^3(\mathcal{A})_r & \xrightarrow{\vec\nabla\varphi} & \mathfrak{X}^2(\mathcal{A})_{r+\varpi(\varphi)} & & \\
 & & \downarrow \vec\nabla & & \downarrow \vec\nabla\times & & \\
\mathfrak{X}^3(\mathcal{A})_{r-\varpi(\varphi)} & \xrightarrow{\vec\nabla\varphi} & \mathfrak{X}^2(\mathcal{A})_r & \xrightarrow{\times\vec\nabla\varphi} & \mathfrak{X}^1(\mathcal{A})_{r+\varpi(\varphi)} & \xrightarrow{\cdot\vec\nabla\varphi} & \mathfrak{X}^0(\mathcal{A})_{r+2\varpi(\varphi)} \\
\downarrow\vec\nabla & & \downarrow\vec\nabla\times & & \downarrow \mathrm{Div} & & \\
\mathfrak{X}^2(\mathcal{A})_{r-\varpi(\varphi)} & \xrightarrow{\times\vec\nabla\varphi} & \mathfrak{X}^1(\mathcal{A})_r & \xrightarrow{\cdot\vec\nabla\varphi} & \mathfrak{X}^0(\mathcal{A})_{r+\varpi(\varphi)} & &
\end{array}$$

Proof. This proposition has already been proved in Propositions 3.1 and 3.5. □

Remark 3.8. If $\varphi \in \mathcal{A}$ is a weight homogeneous polynomial without square factor then the first part of the Koszul complex $\mathcal{A} \xrightarrow{\vec\nabla\varphi} \mathcal{A}^3 \xrightarrow{\times\vec\nabla\varphi} \mathcal{A}^3$ is exact, but the second part

$$\mathcal{A}^3 \xrightarrow{\times\vec\nabla\varphi} \mathcal{A}^3 \xrightarrow{\cdot\vec\nabla\varphi} \mathcal{A} \tag{3.11}$$

need not be exact if the weight homogeneous polynomial φ does not admit the origin as an isolated singularity any more. For example, let $\varphi = xyz \in \mathcal{A}$. The polynomial φ is square free but the origin is not an isolated singularity for φ. Then, the element $\vec F = (x, y, -2z) \in \mathcal{A}$ satisfies the equation $\vec F \cdot \vec\nabla\varphi = \vec 0$ but, by an argument of degree, there is no element $\vec G \in \mathcal{A}^3$ such that $\vec F = \vec G \times \vec\nabla\varphi$.

We will often apply Proposition 3.6 directly but sometimes, we will use it in terms of the following corollary.

Corollary 3.9. *Let $\varphi \in \mathcal{A}$ be a weight homogeneous polynomial with an isolated singularity and let $\vec H \in \mathcal{A}^3$. If $(\vec\nabla \times \vec H) \cdot \vec\nabla\varphi = 0$, then there exist $F, G \in \mathcal{A}$ such that $\vec H = \vec\nabla F + G\vec\nabla\varphi$.*

Proof. According to the diagram in Remark 3.7, the operator $\vec H \mapsto (\vec\nabla \times \vec H) \cdot \vec\nabla\varphi$, considered as a map between $\mathfrak{X}^2(\mathcal{A})$ and $\mathfrak{X}^0(\mathcal{A})$, is a weight homogeneous operator of degree $\varpi(\varphi)$. Therefore, it suffices to prove the result for an element $\vec H \in \mathfrak{X}^2(\mathcal{A})_r$, with $r \in \mathbf{Z}$. If $(\vec\nabla \times \vec H) \cdot \vec\nabla\varphi = 0$ then, by Proposition 3.6, there exists $\vec K \in \mathcal{A}^3$ such that $\vec\nabla \times \vec H = \vec K \times \vec\nabla\varphi$. In view of Remark 3.7, $\vec K$ can be chosen in $\mathfrak{X}^2(\mathcal{A})_{r-\varpi(\varphi)}$. Summarizing, we have to prove that an equation of the type:

$$\vec\nabla \times \vec H = \vec K \times \vec\nabla\varphi, \quad \vec H \in \mathfrak{X}^2(\mathcal{A})_r,\ \vec K \in \mathfrak{X}^2(\mathcal{A})_{r-\varpi(\varphi)} \tag{3.12}$$

implies that $\vec H = \vec\nabla F + G\vec\nabla\varphi$, with $F, G \in \mathcal{A}$.

We will do this by induction on $r \in \mathbf{Z}$, by proving the result directly for all $r < \varpi(\varphi) - \varpi^{[2]}$, with $\varpi^{[2]} := \max\{\varpi_1 + \varpi_2, \varpi_1 + \varpi_3, \varpi_2 + \varpi_3\}$, where the integers $\varpi_1, \varpi_2, \varpi_3$ are the weights of the variables x, y, z.

If $r < \varpi(\varphi) - \varpi^{[2]}$ then, according to the decompositions in (3.7), we have $\mathfrak{X}^2(\mathcal{A})_{r-\varpi(\varphi)} = \{0\}$ so that the equality (3.12) leads to $\vec\nabla \times \vec H = \vec 0$. Using Proposition 3.6, we obtain $\vec H = \vec\nabla F$, with $F \in \mathcal{A}$, as required.

Let $r' \geq \varpi(\varphi) - \varpi^{[2]}$ and assume that (3.12) implies, for all $r < r'$, the existence of $F, G \in \mathcal{A}$ such that $\vec{H} = \vec{\nabla} F + G\vec{\nabla}\varphi$. Let us suppose that an element $\vec{L} \in \mathfrak{X}^2(\mathcal{A})_{r'}$ satisfies an equation like in (3.12), namely, suppose that there exists $\vec{H} \in \mathfrak{X}^2(\mathcal{A})_{r'-\varpi(\varphi)}$ such that

$$\vec{\nabla} \times \vec{L} = \vec{H} \times \vec{\nabla}\varphi. \tag{3.13}$$

Then, $\vec{H}$ satisfies (3.12), with $r = r' - \varpi(\varphi)$. Indeed, computing the divergence of both summands of (3.13) gives $(\vec{\nabla} \times \vec{H}) \cdot \vec{\nabla}\varphi = 0$ and using Proposition 3.6 once again leads to the existence of $\vec{K} \in \mathfrak{X}^2(\mathcal{A})_{r'-2\varpi(\varphi)}$ such that we have $\vec{\nabla} \times \vec{H} = \vec{K} \times \vec{\nabla}\varphi$. By induction hypothesis, there exist $F, G \in \mathcal{A}$ such that $\vec{H} = \vec{\nabla} F + G\vec{\nabla}\varphi$. Then, using Formula (3.1), we obtain

$$\vec{\nabla} \times \vec{L} = \vec{H} \times \vec{\nabla}\varphi = \vec{\nabla} F \times \vec{\nabla}\varphi = \vec{\nabla} \times (F\vec{\nabla}\varphi).$$

We can now conclude with Proposition 3.6 that there exists $F' \in \mathcal{A}$ such that $\vec{L} - F\vec{\nabla}\varphi = \vec{\nabla} F'$. Hence the result. $\square$

3.2 Cohomology of $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$

In this section, we will determine the Poisson cohomology spaces of the Poisson variety $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$, where $\varphi \in \mathcal{A} = \mathbf{F}[x, y, z]$ is a weight homogeneous polynomial, with an isolated singularity (at the origin) and $\{\cdot\,,\cdot\}_\varphi$ is the Poisson bracket defined in the paragraph 2.1.3.

3.2.1 Poisson complex of $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$

Let us recall that for the polynomial Poisson algebra $\mathcal{A} = \mathbf{F}[x, y, z]$, we have the following isomorphisms (see paragraph 3.1.1):

$$\begin{aligned} \mathfrak{X}^1(\mathcal{A}) &\longrightarrow \mathcal{A}^3 \\ \mathcal{V} &\longmapsto (\mathcal{V}[x], \mathcal{V}[y], \mathcal{V}[z]); \end{aligned}$$

$$\begin{aligned} \mathfrak{X}^2(\mathcal{A}) &\longrightarrow \mathcal{A}^3 \\ \mathcal{W} &\longmapsto (\mathcal{W}[y, z], \mathcal{W}[z, x], \mathcal{W}[x, y]); \end{aligned}$$

$$\begin{aligned} \mathfrak{X}^3(\mathcal{A}) &\longrightarrow \mathcal{A} \\ \mathcal{Z} &\longmapsto \mathcal{Z}[x, y, z]. \end{aligned}$$

For example, with the notations of the paragraph 3.1.1, the skew-symmetric biderivation $\{\cdot\,,\cdot\}_\varphi$ is identified with the element $\vec{\nabla}\varphi$ of $\mathcal{A}^3$.

Each of the Poisson coboundary operators associated to the Poisson algebra $(\mathcal{A}, \{\cdot\,,\cdot\}_\varphi)$ (see paragraph 2.2.1), denoted by δ^k_φ, can now be written in a compact form:

$$\begin{aligned}
&\delta^0_\varphi(F) = \vec{\nabla} F \times \vec{\nabla}\varphi, \quad \text{ for } F \in \mathcal{A} \simeq \mathfrak{X}^0(\mathcal{A}),\\
&\delta^1_\varphi(\vec{F}) = -\vec{\nabla}(\vec{F}\cdot\vec{\nabla}\varphi) + \operatorname{Div}(\vec{F})\vec{\nabla}\varphi, \quad \text{ for } \vec{F} \in \mathcal{A}^3 \simeq \mathfrak{X}^1(\mathcal{A}),\\
&\delta^2_\varphi(\vec{F}) = -\vec{\nabla}\varphi\cdot(\vec{\nabla}\times\vec{F}) = -\operatorname{Div}(\vec{F}\times\vec{\nabla}\varphi), \quad \text{ for } \vec{F} \in \mathcal{A}^3 \simeq \mathfrak{X}^2(\mathcal{A}).
\end{aligned} \tag{3.14}$$

Let us for example determine $\delta^0_\varphi(F)$ as an element of $\mathcal{A}^3$, for an arbitrary $F \in \mathcal{A}$. We will identify $\delta^0_\varphi(F)$ with the triplet $\big(\delta^0_\varphi(F)[x], \delta^0_\varphi(F)[y], \delta^0_\varphi(F)[z]\big)$. We have, according to Formula (2.12),

$$\delta^0_\varphi(F)[x] = \{x, F\}_\varphi = -\frac{\partial\varphi}{\partial y}\frac{\partial F}{\partial z} + \frac{\partial\varphi}{\partial z}\frac{\partial F}{\partial y} = \left(\vec{\nabla} F \times \vec{\nabla}\varphi\right)_1,$$

the first component of the element $\vec{\nabla} F \times \vec{\nabla}\varphi \in \mathcal{A}^3$, and analogous equalities for $\delta^0_\varphi(F)[y]$ and $\delta^0_\varphi(F)[z]$, so that we obtain the desired formula for $\delta^0_\varphi(F)$.

As a consequence, the Poisson cohomology spaces of $(\mathcal{A}, \{\cdot\,,\cdot\}_\varphi)$, denoted by $H^k(\mathcal{A}, \varphi)$, take the following forms

$$\begin{aligned}
H^0(\mathcal{A},\varphi) &= \operatorname{Cas}(\mathcal{A},\varphi) \simeq \{F \in \mathcal{A} \mid \vec{\nabla} F \times \vec{\nabla}\varphi = \vec{0}\,\},\\
H^1(\mathcal{A},\varphi) &\simeq \frac{\{\vec{F} \in \mathcal{A}^3 \mid -\vec{\nabla}(\vec{F}\cdot\vec{\nabla}\varphi) + \operatorname{Div}(\vec{F})\vec{\nabla}\varphi = \vec{0}\,\}}{\{\vec{\nabla} F \times \vec{\nabla}\varphi \mid F \in \mathcal{A}\}},\\
H^2(\mathcal{A},\varphi) &\simeq \frac{\{\vec{F} \in \mathcal{A}^3 \mid \vec{\nabla}\varphi\cdot(\vec{\nabla}\times\vec{F}) = 0\}}{\{-\vec{\nabla}(\vec{F}\cdot\vec{\nabla}\varphi) + \operatorname{Div}(\vec{F})\vec{\nabla}\varphi \mid \vec{F} \in \mathcal{A}^3\}},\\
H^3(\mathcal{A},\varphi) &\simeq \frac{\mathcal{A}}{\{\vec{\nabla}\varphi\cdot(\vec{\nabla}\times\vec{F}) \mid \vec{F} \in \mathcal{A}^3\}}
\end{aligned}$$

and we denote by $B^k(\mathcal{A},\varphi)$ (respectively, $Z^k(\mathcal{A},\varphi)$) the space of all k-coboundaries (respectively, k-cocycles) of $(\mathcal{A}, \{\cdot\,,\cdot\}_\varphi)$.

One of our purposes is to determine the Poisson cohomology of $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$ when $\varphi \in \mathcal{A}$ is weight homogeneous with an isolated singularity. The weight homogeneity of φ will be essential for the computation of these spaces. It implies indeed, among other things, that each of the coboundary operators δ^k_φ is weight homogeneous of the same degree $N(\varphi) := \varpi(\varphi) - |\varpi|$, as can be seen from (3.14). That is to say, we have:

$$P \in \mathfrak{X}^k(\mathcal{A})_i \Rightarrow \delta^k_\varphi(P) \in \mathfrak{X}^{k+1}(\mathcal{A})_{i+N(\varphi)}.$$

If $P \in \mathfrak{X}^k(\mathcal{A})$ is a cocycle, then each of its weight homogeneous components will be a cocycle. In the same way, if $P \in \mathfrak{X}^k(\mathcal{A})$ is a coboundary then each of its weight homogeneous components will be a coboundary. Moreover, if $P \in \mathfrak{X}^k(\mathcal{A})$ is a weight homogeneous coboundary, it is the coboundary of a weight homogeneous element in $\mathfrak{X}^{k-1}(\mathcal{A})$.

Remark 3.10. If $\varphi \in \mathcal{A}$ is a weight homogeneous polynomial with an isolated singularity, then $\varpi(\varphi) - \varpi_i > 0$, for $i = 1, 2, 3$ (where $\varpi(\varphi)$ is still the (weighted) degree of φ and $\varpi_1, \varpi_2, \varpi_3$ are the weights of the variables x, y, z), and in particular, $\varpi(\varphi) > 1$.

3.2.2 The space $H^0(\mathcal{A}, \varphi)$

A precise description of the 0-th Poisson cohomology space, which is also the algebra of the Casimirs, is given in the following proposition.

Proposition 3.11. *If $\varphi \in \mathcal{A}$ is weight homogeneous with an isolated singularity, then the zeroth Poisson cohomology space of $(\mathcal{A}, \{\cdot\,,\cdot\}_\varphi)$ is given by*

$$H^0(\mathcal{A}, \varphi) = \mathrm{Cas}(\mathcal{A}, \varphi) \simeq \bigoplus_{i \in \mathbf{N}} \mathbf{F}\varphi^i.$$

Proof. Let $F \in \mathcal{A} - \{0\}$ be a weight homogeneous 0-cocycle, thus satisfying $\delta^0_\varphi(F) = \vec{\nabla} F \times \vec{\nabla}\varphi = \vec{0}$. Write F as $F = H\varphi^r$, where $r \in \mathbf{N}$ and where $H \in \mathcal{A} - \{0\}$ is a polynomial that is not divisible by φ. We have $\vec{\nabla} F = \varphi^r \vec{\nabla} H + rH\varphi^{r-1}\vec{\nabla}\varphi$, so $\vec{\nabla} H \times \vec{\nabla}\varphi = \vec{0}$. Proposition 3.6 implies the existence of $G \in \mathcal{A}$ such that $\vec{\nabla} H = G\vec{\nabla}\varphi$. Since H and φ are weight homogeneous and in view of Euler's Formula (3.5),

$$\varpi(H)\, H = \vec{\nabla} H \cdot \vec{e}_\varpi = G\vec{\nabla}\varphi \cdot \vec{e}_\varpi = \varpi(\varphi)\, G\varphi,$$

so $\varpi(H) = 0$, as H is not divisible by φ. Thus $H \in \mathbf{F}$ and $F = H\varphi^r \in \bigoplus_{i\in\mathbf{N}} \mathbf{F}\varphi^i$. Conversely, it is clear that $\delta^0_\varphi(\varphi^r) = \vec{\nabla}(\varphi^r) \times \vec{\nabla}\varphi = \vec{0}$, for any $r \in \mathbf{N}$. □

Remark 3.12. According to Remark 3.8, if $\varphi \in \mathcal{A}$ is a weight homogeneous polynomial without square factor but φ is not necessarily with an isolated singularity, then the first part of the Koszul complex is still exact, so Proposition 3.11 is also valid for this more general class of polynomials. However, if φ has a square factor, the result is not true anymore. For example, if $\varphi = \psi^r$ with $r \geq 2$ and $\psi \in \mathcal{A}$ a weight homogeneous polynomial without square factor, then $H^0(\mathcal{A}, \varphi) \simeq H^0(\mathcal{A}, \psi) \simeq \bigoplus_{i\in\mathbf{N}} \mathbf{F}\psi^i$ so that $H^0(\mathcal{A}, \varphi) \not\simeq \bigoplus_{i\in\mathbf{N}} \mathbf{F}\varphi^i$.

3.2.3 The space $H^1(\mathcal{A}, \varphi)$

We first prove a result which will be useful to determine $H^1(\mathcal{A}, \varphi)$.

Lemma 3.13. *Let $\varphi \in \mathcal{A}$ be a weight homogeneous polynomial with an isolated singularity and $\vec{G} \in \mathcal{A}^3$. Suppose that there exist $r \in \mathbf{N}$ and $\alpha \in \mathbf{F}$ such that*

$$\begin{cases} \vec{G} \cdot \vec{\nabla}\varphi = 0, \\ \mathrm{Div}(\vec{G}) = \alpha\varphi^r. \end{cases} \tag{3.15}$$

Then $\alpha = 0$ (equivalently $\mathrm{Div}(\vec{G}) = 0$).

Proof. According to Remark 3.7, the operator $\vec{G} \mapsto (\vec{G} \cdot \vec{\nabla}\varphi, \mathrm{Div}(\vec{G}))$ (from $\mathcal{A}^3$ to $\mathcal{A}^2$) restricts for any $d \in \mathbf{Z}$ to an operator between $\mathfrak{X}^1(\mathcal{A})_d$ and $\mathfrak{X}^0(\mathcal{A})_{d+\varpi(\varphi)} \times \mathfrak{X}^0(\mathcal{A})_d$. Therefore it suffices to prove the lemma for an element $\vec{G} \in \mathfrak{X}^1(\mathcal{A})_d$, with $d \in \mathbf{Z}$. Suppose that such an element $\vec{G}$ satifies (3.15), then, according to Proposition 3.6, the first equation implies that there exists $\vec{K} \in \mathfrak{X}^2(\mathcal{A})_{d-\varpi(\varphi)}$, such that $\vec{G} = \vec{K} \times \vec{\nabla}\varphi$. We will apply induction on $r \in \mathbf{N}$. First, if $r = 0$, then, according to Formula (3.3), $\alpha = \mathrm{Div}(\vec{G}) = \mathrm{Div}(\vec{K} \times \vec{\nabla}\varphi) = (\vec{\nabla} \times \vec{K}) \cdot \vec{\nabla}\varphi$, so that $\alpha = 0$, for degree reasons.

Assume now that for some fixed $r \geq 0$, any $\vec{G}$ that satisfies (3.15) is divergence free. Suppose that $\vec{H} \in \mathcal{A}^3$ satisfies $\vec{H} \cdot \vec{\nabla}\varphi = 0$ and $\mathrm{Div}(\vec{H}) = \alpha'\varphi^{r+1}$, for some $\alpha' \in \mathbf{F}$. Writing $\vec{H} = \vec{K} \times \vec{\nabla}\varphi$, the Formulas (3.3), (3.5) and (3.6) show that $\vec{G} := \vec{\nabla} \times \vec{K} - \frac{\alpha'}{\varpi(\varphi)}\varphi^r \vec{e}_\varpi$ satisfies (3.15), with $\alpha = -\alpha'(\varpi(\varphi)r + |\varpi|)/\varpi(\varphi)$, so that, by induction hypothesis, $0 = \alpha = -\alpha'(\varpi(\varphi)r + |\varpi|)/\varpi(\varphi)$. It follows that $\alpha' = 0$. □

Now, we can give the main result of this Section. We recall that $|\varpi|$ is the sum of the weights of the three variables x, y, z.

Proposition 3.14. *If $\varphi \in \mathcal{A}$ is weight homogeneous polynomial with an isolated singularity, then the first Poisson cohomology space of $(\mathcal{A}, \{\cdot\,,\cdot\}_\varphi)$ is a free module over $\mathrm{Cas}(\mathcal{A}, \varphi)$, given by:*

$$H^1(\mathcal{A}, \varphi) \simeq \begin{cases} \{0\} & \text{if } \varpi(\varphi) \neq |\varpi|; \\ \mathrm{Cas}(\mathcal{A}, \varphi)\, \vec{e}_\varpi = \bigoplus_{i \in \mathbf{N}} \mathbf{F}\varphi^i\, \vec{e}_\varpi & \text{if } \varpi(\varphi) = |\varpi|. \end{cases}$$

Proof. Let $\vec{F} \in \mathfrak{X}^1(\mathcal{A})$ be a non zero element of $Z^1(\mathcal{A}, \varphi)$, that is to say, $\vec{F} \in \mathcal{A}^3$ satisfies the equation:

$$\vec{\nabla}(\vec{F} \cdot \vec{\nabla}\varphi) = \mathrm{Div}(\vec{F})\, \vec{\nabla}\varphi. \tag{3.16}$$

According to Remark 3.7, we suppose that $\vec{F}$ is weight homogeneous. Our purpose is to write $\vec{F} = \vec{\nabla}K \times \vec{\nabla}\varphi + \frac{c}{\varpi(\varphi)}\varphi^r \vec{e}_\varpi \in B^1(\mathcal{A}, \varphi) + \bigoplus_{i \in \mathbf{N}} \mathbf{F}\varphi^i\, \vec{e}_\varpi$, where $c = 0$ if $\varpi(\varphi) \neq |\varpi|$ and c need not be 0 otherwise. Our proof will be divided in three parts.

1. First, using cocycle condition (3.16), we find an element $\vec{G} \in \mathcal{A}^3$ which satisfies the equations (3.15). This equality implies indeed that $\delta^0_\varphi(\vec{F} \cdot \vec{\nabla}\varphi) =$

$\vec{\nabla}(\vec{F} \cdot \vec{\nabla}\varphi) \times \vec{\nabla}\varphi = \vec{0}$, so that the weight homogeneous element $\vec{F} \cdot \vec{\nabla}\varphi$ of $\mathcal{A}$ is a Casimir. According to Proposition 3.11, there exist $c \in \mathbf{F}$ and $r \in \mathbf{N}$ such that $\vec{F} \cdot \vec{\nabla}\varphi = c\varphi^{r+1}$. Using Equation (3.16) once more, we obtain $\mathrm{Div}(\vec{F}) = c(r+1)\varphi^r$. Letting $\vec{G} := \vec{F} - \frac{c}{\varpi(\varphi)}\varphi^r \vec{e}_\varpi$, Formulas (3.5) and (3.6) imply that $\vec{G}$ satisfies (3.15), where $\alpha = c(1 - \frac{|\varpi|}{\varpi(\varphi)})$. Lemma 3.13 leads to

$$\begin{cases} \mathrm{Div}(\vec{G}) = 0, \ \vec{G} \cdot \vec{\nabla}\varphi = 0, \\ 0 = c\left(1 - \frac{|\varpi|}{\varpi(\varphi)}\right). \end{cases}$$

2. Now, we will show that if $\vec{G} \in \mathcal{A}^3$ satisfies $\mathrm{Div}(\vec{G}) = 0$ and $\vec{G} \cdot \vec{\nabla}\varphi = 0$, then $\vec{G} \in B^1(\mathcal{A}, \varphi)$. Let $\vec{G}$ be a such element. As $\vec{G} \cdot \vec{\nabla}\varphi = 0$, Proposition 3.6 implies the existence of an element $\vec{H} \in \mathcal{A}^3$ such that $\vec{G} = \vec{H} \times \vec{\nabla}\varphi$. Moreover, we have

$$0 = \mathrm{Div}(\vec{G}) = \mathrm{Div}(\vec{H} \times \vec{\nabla}\varphi) = (\vec{\nabla} \times \vec{H}) \cdot \vec{\nabla}\varphi.$$

Corollary 3.9 leads now to the existence of elements $K, L \in \mathcal{A}$ such that $\vec{H} = \vec{\nabla}K + L\vec{\nabla}\varphi$, so that $\vec{G} = \vec{\nabla}K \times \vec{\nabla}\varphi = \delta^0_\varphi(K) \in B^1(\mathcal{A}, \varphi)$.

3. The first two parts of this proof lead to the existence of $K \in \mathcal{A}$ and $c \in \mathbf{F}$ such that

$$\begin{cases} \vec{F} = \vec{\nabla}K \times \vec{\nabla}\varphi + \frac{c}{\varpi(\varphi)}\varphi^r \vec{e}_\varpi, \\ 0 = c\left(1 - \frac{|\varpi|}{\varpi(\varphi)}\right). \end{cases} \tag{3.17}$$

Now, we have to consider two cases: $\varpi(\varphi) \neq |\varpi|$ and $\varpi(\varphi) = |\varpi|$.

- If $\varpi(\varphi) \neq |\varpi|$ then $c = 0$ and $\vec{F} = \vec{\nabla}K \times \vec{\nabla}\varphi = \delta^0_\varphi(K) \in B^1(\mathcal{A}, \varphi)$. Thus, when $\varpi(\varphi) \neq |\varpi|$, then $H^1(\mathcal{A}, \varphi) \simeq \{0\}$.
- Now, suppose that $\varpi(\varphi) = |\varpi|$, then (3.17) leads to $Z^1(\mathcal{A}, \varphi) \subseteq B^1(\mathcal{A}, \varphi) + \bigoplus_{i \in \mathbf{N}} \mathbf{F}\varphi^i \vec{e}_\varpi$. Conversely, for any $i \in \mathbf{N}$, Formulas (3.5) and (3.6) lead to $\delta^1_\varphi(\varphi^i \vec{e}_\varpi) = (|\varpi| - \varpi(\varphi))\varphi^i \vec{\nabla}\varphi = 0$. So that

$$Z^1(\mathcal{A}, \varphi) = B^1(\mathcal{A}, \varphi) + \bigoplus_{i \in \mathbf{N}} \mathbf{F}\varphi^i \vec{e}_\varpi.$$

Let us show that this sum is a direct one. It suffices to consider a weight homogeneous element $\alpha\varphi^i \vec{e}_\varpi \in B^1(\mathcal{A}, \varphi)$, $\alpha \in \mathbf{F}$, $i \in \mathbf{N}$. It means that there exists $K \in \mathcal{A}$ such that $\alpha\varphi^i \vec{e}_\varpi = \vec{\nabla}K \times \vec{\nabla}\varphi$. Then (3.3) and (3.6) lead to (we suppose $\varpi(\varphi) = |\varpi|$)

$$0 = \mathrm{Div}(\vec{\nabla}K \times \vec{\nabla}\varphi) = \mathrm{Div}(\alpha\varphi^i \vec{e}_\varpi) = \alpha|\varpi|(i+1)\varphi^i,$$

therefore $\alpha = 0$ and the sum $B^1(\mathcal{A}, \varphi) \oplus \bigoplus_{i \in \mathbf{N}} \mathbf{F}\varphi^i \vec{e}_\varpi$ is direct. Thus, when $\varpi(\varphi) = |\varpi|$, then $H^1(\mathcal{A}, \varphi) \simeq \bigoplus_{i \in \mathbf{N}} \mathbf{F}\varphi^i \vec{e}_\varpi$. □

Remark 3.15. We see that the case $\varpi(\varphi) = |\varpi|$ is particular. When φ is homogeneous (i.e. weight homogeneous with $\varpi_1 = \varpi_2 = \varpi_3 = 1$), it is the case where the degree of φ is three, that is to say, where φ is a cubic polynomial and $\{\cdot,\cdot\}_\varphi$ is a quadratic Poisson bracket.

3.2.4 The space $H^3(\mathcal{A}, \varphi)$

Now, we give the third Poisson cohomology space of $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$, where $\varphi \in \mathcal{A} = \mathbf{F}[x, y, z]$ is weight homogeneous with an isolated singularity. Recall that, in this case,

$$\mathcal{A}_{sing} = \mathbf{F}[x, y, z]/\langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z} \rangle$$

is a finite dimensional $\mathbf{F}$-vector space, whose dimension is the Milnor number, denoted by μ. Let $u_0 = 1, u_1, \ldots, u_{\mu-1}$ be weight homogeneous elements of $\mathcal{A}$, such that their images in $\mathcal{A}_{sing}$ give a $\mathbf{F}$-basis of $\mathcal{A}_{sing}$. We still denote by $|\varpi| = \varpi_1 + \varpi_2 + \varpi_3$ the sum of the weights of x, y and z.

Proposition 3.16. *If $\varphi \in \mathcal{A} = \mathbf{F}[x, y, z]$ is weight homogeneous with an isolated singularity, then the third cohomology space $H^3(\mathcal{A}, \varphi)$ is the free* $\mathrm{Cas}(\mathcal{A}, \varphi)$-*module:*

$$H^3(\mathcal{A}, \varphi) \simeq \bigoplus_{j=0}^{\mu-1} \mathrm{Cas}(\mathcal{A}, \varphi)\, u_j \simeq \mathrm{Cas}(\mathcal{A}, \varphi) \otimes_{\mathbf{F}} \mathcal{A}_{sing}.$$

Proof. Let $F \in \mathcal{A}_d \simeq \mathfrak{X}^3(\mathcal{A})_{d-|\varpi|}$ be a weight homogeneous polynomial of degree $d \in \mathbf{N}$.

1. We first show that there exist $\vec{G} \in \mathcal{A}^3$, $N \in \mathbf{N}$ and elements $\lambda_{i,j} \in \mathbf{F}$, where $0 \leq i \leq N$ and $0 \leq j \leq \mu - 1$, such that:

$$\begin{aligned} F &= \vec{\nabla}\varphi \cdot (\vec{\nabla} \times \vec{G}) + \sum_{i=0}^{N} \sum_{j=0}^{\mu-1} \lambda_{i,j} \varphi^i u_j \\ &\in B^3(\mathcal{A}, \varphi) + \sum_{\substack{k \in \mathbf{N} \\ 0 \leq j \leq \mu-1}} \mathbf{F} \varphi^k u_j. \end{aligned} \tag{3.18}$$

Let $\varpi^{[1]} := \max(\varpi_1, \varpi_2, \varpi_3)$. We apply induction on d, proving directly the result for $d \leq \varpi(\varphi) - \varpi^{[1]}$ (this is not an empty case, as can be seen from Remark 3.10, for example, it contains the case $F \in \mathbf{F}$). By definition of the elements $u_0, \ldots, u_{\mu-1}$, we have:

$$F = \vec{\nabla}\varphi \cdot \vec{L} + \sum_{j=0}^{\mu-1} \alpha_j u_j, \tag{3.19}$$

where $\vec{L} \in \mathfrak{X}^1(\mathcal{A})_{d-\varpi(\varphi)}$ and $\alpha_0, \ldots, \alpha_{\mu-1} \in \mathbf{F}$.

If $d \leq \varpi(\varphi) - \varpi^{[1]}$ then the correspondences (3.7) imply that $\vec{L}$ is an element (a, b, c) of $\mathbf{F}^3$ so that F is indeed of the form (3.18), with $\vec{G} = (bz, cx, ay)$, $N = 0$ and $\lambda_{0,j} = \alpha_j$.

Now, suppose that $d > \varpi(\varphi) - \varpi^{[1]}$ and that any weight homogeneous polynomial of degree at most $d-1$ is of the form (3.18). Let us consider the decomposition (3.19) for F of degree d. Proposition 3.6 implies that there exists $\vec{G} \in \mathcal{A}^3$ such that:

$$\vec{L} - \frac{\mathrm{Div}(\vec{L})}{d - \varpi(\varphi) + |\varpi|}\vec{e}_\varpi = \vec{\nabla} \times \vec{G}, \tag{3.20}$$

since $\mathrm{Div}\left(\vec{L} - \frac{\mathrm{Div}(\vec{L})}{d-\varpi(\varphi)+|\varpi|}\vec{e}_\varpi\right) = 0$, as follows from $\varpi(\mathrm{Div}(\vec{L})) = d - \varpi(\varphi)$ and (3.6).

Using the induction hypothesis on $\mathrm{Div}(\vec{L})$, we obtain the existence of a $\vec{K} \in \mathcal{A}^3$, satisfying

$$\mathrm{Div}(\vec{L}) \in \vec{\nabla}\varphi \cdot \left(\vec{\nabla} \times \vec{K}\right) + \sum_{\substack{k \in \mathbf{N} \\ 0 \le j \le \mu - 1}} \mathbf{F}\varphi^k u_j.$$

Then, (3.19), (3.20) and the Euler formula (3.5) imply:

$$\begin{aligned} F &\in \frac{\varpi(\varphi)}{d - \varpi(\varphi) + |\varpi|}\varphi\vec{\nabla}\varphi \cdot \left(\vec{\nabla} \times \vec{K}\right) + \sum_{\substack{l \in \mathbf{N} \\ 0 \le j \le \mu - 1}} \mathbf{F}\varphi^l u_j + \left(\vec{\nabla} \times \vec{G}\right) \cdot \vec{\nabla}\varphi \\ &\in \vec{\nabla}\varphi \cdot \left(\vec{\nabla} \times \left(\frac{\varpi(\varphi)}{d - \varpi(\varphi) + |\varpi|}\varphi\vec{K} + \vec{G}\right)\right) + \sum_{\substack{l \in \mathbf{N} \\ 0 \le j \le \mu - 1}} \mathbf{F}\varphi^l u_j, \end{aligned}$$

because of Formula (3.1). We have then obtained an equation of the form (3.18).

2. So, we have already obtained that

$$\begin{aligned} \mathcal{A} &= \{\vec{\nabla}\varphi \cdot (\vec{\nabla} \times \vec{L}) \mid \vec{L} \in \mathcal{A}^3\} + \sum_{j=0}^{\mu-1} \mathrm{Cas}(\mathcal{A}, \varphi) u_j \\ &= B^3(\mathcal{A}, \varphi) + \sum_{j=0}^{\mu-1} \mathrm{Cas}(\mathcal{A}, \varphi) u_j. \end{aligned} \tag{3.21}$$

and it suffices to show that this sum is direct in $\mathcal{A} \simeq \mathfrak{X}^3(\mathcal{A})$.

We suppose the contrary. This allows us to consider the smallest integer $N_0 \in \mathbf{N}$ such that we have an equation of the form:

$$\sum_{i=N_0}^{N} \sum_{j=0}^{\mu-1} \lambda_{i,j}\varphi^i u_j = \vec{\nabla}\varphi \cdot (\vec{\nabla} \times \vec{G}) = -\delta^2_\varphi(\vec{G}), \tag{3.22}$$

with $\vec{G} \in \mathcal{A}^3$, $N \ge N_0$ and $\lambda_{i,j} \in \mathbf{F}$ (for $N_0 \le i \le N$ and $0 \le j \le \mu - 1$) and $\lambda_{N_0, j_0} \ne 0$, for some $0 \le j_0 \le \mu - 1$. We will show that this hypothesis leads to a contradiction.

First, suppose that $N_0 = 0$, then, according to Euler's Formula (3.5),

$$\sum_{j=0}^{\mu-1}\lambda_{0,j}u_j = -\sum_{i=1}^{N}\sum_{j=0}^{\mu-1}\lambda_{i,j}\varphi^i u_j + \vec{\nabla}\varphi\cdot(\vec{\nabla}\times\vec{G}) \in \langle\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial y},\frac{\partial\varphi}{\partial z}\rangle$$

and the definition of the u_j implies that $\lambda_{0,j} = 0$ for all $0 \le j \le \mu - 1$, which contradicts the hypothesis $\lambda_{N_0,j_0} \neq 0$.

So we suppose that $N_0 > 0$, using Euler's Formula (3.5), the equation (3.22) can be written as $\vec{\nabla}\varphi\cdot\left(\sum_{i=N_0}^{N}\sum_{j=0}^{\mu-1}\frac{\lambda_{i,j}}{\varpi(\varphi)}\varphi^{i-1}u_j\vec{e}_\varpi\right) = \vec{\nabla}\varphi\cdot(\vec{\nabla}\times\vec{G})$. Proposition 3.6 implies that there exists $\vec{H}\in\mathcal{A}^3$ such that:

$$\sum_{i=N_0}^{N}\sum_{j=0}^{\mu-1}\frac{\lambda_{i,j}}{\varpi(\varphi)}\varphi^{i-1}u_j\vec{e}_\varpi = \vec{\nabla}\times\vec{G}+\vec{H}\times\vec{\nabla}\varphi.$$

The divergence of both sides of this equality and Formula (3.6) give:

$$\sum_{i=N_1}^{N}\sum_{j=0}^{\mu-1}\lambda'_{i,j}\varphi^i u_j = (\vec{\nabla}\times\vec{H})\cdot\vec{\nabla}\varphi = -\delta^2_\varphi(\vec{H}),$$

where $\lambda'_{i,j} = \frac{\lambda_{i+1,j}}{\varpi(\varphi)}(\varpi(\varphi)i+\varpi(u_j)+|\varpi|)$ and $N_1 = N_0-1$. So, we have obtained an equation of the form (3.22), with $N_1 < N_0$ and $\lambda'_{N_1,j_0} \neq 0$. This fact contradicts the hypothesis and we conclude that the sum (3.21) is direct. The description of $H^3(\mathcal{A},\varphi)$ follows. □

Remark 3.17. Proposition 3.16 permits us to give a short proof of Lemma 3.13. Indeed, let $\vec{G}\in\mathcal{A}^3$ satisfying the system (3.15), with $\alpha\in\mathbf{F}$ and $r\in\mathbf{N}$. The first equation $\vec{G}\cdot\vec{\nabla}\varphi = 0$ implies, according to the exactness of the Koszul diagram (Proposition 3.5), that there exists $\vec{K}\in\mathfrak{X}^2(\mathcal{A})$, such that $\vec{G} = \vec{K}\times\vec{\nabla}\varphi$. Then the second equation becomes:

$$\alpha\varphi^r = \mathrm{Div}(\vec{G}) = \mathrm{Div}(\vec{K}\times\vec{\nabla}\varphi) = (\vec{\nabla}\times\vec{K})\cdot\vec{\nabla}\varphi.$$

We point out that $\alpha\varphi^r \in \mathrm{Cas}(\mathcal{A},\varphi)\,u_0$, so that, according to Proposition 3.16 and the writing of $H^3(\mathcal{A},\varphi)$ in Paragraph 3.2.1, we conclude that $\alpha = 0$, that is the result given in Lemma 3.13.

3.2.5 The space $H^2(\mathcal{A},\varphi)$

Finally, using Proposition 3.16 (and in fact the writing of $H^3(\mathcal{A},\varphi)$), we obtain a $\mathbf{F}$-basis of the second Poisson cohomology space of the algebra $(\mathcal{A},\{\cdot\,,\cdot\}_\varphi)$, when $\varphi\in\mathcal{A} = \mathbf{F}[x,y,z]$ is a weight homogeneous polynomial, with an isolated singularity. We point out that, in Chapter 6, we will obtain another basis of this space that will be more useful for the study of the formal deformations of $\{\cdot\,,\cdot\}_\varphi$, but the basis we give in the following proposition permits one to see easily for example, the free part of $H^2(\mathcal{A},\varphi)$ (as $\mathrm{Cas}(\mathcal{A},\varphi)$-module) and its writing in the special cases $\varpi(\varphi) < |\varpi|$ and $\varpi(\varphi) = |\varpi|$.

Remark 3.18. As $Z^2(\mathcal{A},\varphi) = \{\vec{H} \in \mathcal{A}^3 \mid (\vec{\nabla} \times \vec{H}) \cdot \vec{\nabla}\varphi = 0\}$, Corollary 3.9 leads to the equality

$$Z^2(\mathcal{A},\varphi) = \{\vec{\nabla}F + G\vec{\nabla}\varphi \mid F, G \in \mathcal{A}\}.$$

Proposition 3.19. *If $\varphi \in \mathcal{A} = \mathbf{F}[x,y,z]$ is weight homogeneous with an isolated singularity, then the second Poisson cohomology space of the algebra $(\mathcal{A}, \{\cdot\,,\cdot\}_\varphi)$ is the* $\mathrm{Cas}(\mathcal{A},\varphi)$*-module:*

$$\begin{aligned} H^2(\mathcal{A},\varphi) \simeq & \bigoplus_{\substack{j=1\\ \varpi(u_j)\neq\varpi(\varphi)-|\varpi|}}^{\mu-1} \mathrm{Cas}(\mathcal{A},\varphi)\vec{\nabla}u_j \oplus \bigoplus_{\substack{j=0\\ \varpi(u_j)=\varpi(\varphi)-|\varpi|}}^{\mu-1} \mathrm{Cas}(\mathcal{A},\varphi)u_j\vec{\nabla}\varphi \\ & \oplus \bigoplus_{\substack{j=1\\ \varpi(u_j)=\varpi(\varphi)-|\varpi|}}^{\mu-1} \mathbf{F}\vec{\nabla}u_j, \end{aligned}$$

where the first row gives the free part.

In particular, we have: $H^2(\mathcal{A},\varphi) \simeq \bigoplus_{j=1}^{\mu-1} \mathrm{Cas}(\mathcal{A},\varphi)\vec{\nabla}u_j$, if $\varpi(\varphi) < |\varpi|$ and $H^2(\mathcal{A},\varphi) \simeq \bigoplus_{j=1}^{\mu-1} \mathrm{Cas}(\mathcal{A},\varphi)\vec{\nabla}u_j \oplus \mathrm{Cas}(\mathcal{A},\varphi)\vec{\nabla}\varphi$, when $\varpi(\varphi) = |\varpi|$.

Remark 3.20. We see that the Poisson structure $\{\cdot\,,\cdot\}_\varphi$ will be exact (that is to say a 2-coboundary) if and only if $\varpi(\varphi) \neq |\varpi|$. This fact comes from the equality $\delta^1_\varphi(\vec{e}_\varpi) = -(\varpi(\varphi) - |\varpi|)\vec{\nabla}\varphi$, a consequence of Formulas (3.5) and (3.6).

Remark 3.21. Contrary to the other cohomology spaces, $H^2(\mathcal{A},\varphi)$ is generally not a free $\mathrm{Cas}(\mathcal{A},\varphi)$-module. In fact, using Formulas (3.5) and (3.6), we get:

$$\delta^1_\varphi\left(\varphi^i u_j \vec{e}_\varpi\right) = \left(\varpi(u_j) - \varpi(\varphi) + |\varpi|\right)\varphi^i u_j \vec{\nabla}\varphi - \varpi(\varphi)\varphi^{i+1}\vec{\nabla}u_j. \tag{3.23}$$

This equality, which will be also useful later, explains that we have to distinguish, in the expression of $H^2(\mathcal{A},\varphi)$, the u_j satisfying $\varpi(u_j) = \varpi(\varphi) - |\varpi|$ from the other ones. While $\vec{\nabla}u_j \notin B^2(\mathcal{A},\varphi)$ (for any $j \geq 1$), if j is such that $\varpi(u_j) = \varpi(\varphi) - |\varpi|$ then (3.23) yields that $\varphi^k\vec{\nabla}u_j \in B^2(\mathcal{A},\varphi)$, for all $k \geq 1$, but this is not true when $\varpi(u_j) \neq \varpi(\varphi) - |\varpi|$. This is the reason why $H^2(\mathcal{A},\varphi)$ is not always a free module over $\mathrm{Cas}(\mathcal{A},\varphi)$.

Moreover, for all j satisfying $\varpi(u_j) \neq \varpi(\varphi) - |\varpi|$, (3.23) implies that $\varphi^i u_j \vec{\nabla}\varphi$, $i \geq 0$, can be written as $c\varphi^{i+1}\vec{\nabla}u_j + \delta^1_\varphi\left(c'\varphi^i u_j \vec{e}_\varpi\right)$, with $c, c' \in \mathbf{F} - \{0\}$.

Proof. First, let us show that:

$$\begin{aligned} Z^2(\mathcal{A},\varphi) \simeq B^2(\mathcal{A},\varphi) + & \sum_{\substack{j=1\\ \varpi(u_j)\neq\varpi(\varphi)-|\varpi|}}^{\mu-1} \mathrm{Cas}(\mathcal{A},\varphi)\vec{\nabla}u_j \\ + \sum_{\substack{j=0\\ \varpi(u_j)=\varpi(\varphi)-|\varpi|}}^{\mu-1} & \mathrm{Cas}(\mathcal{A},\varphi)u_j\vec{\nabla}\varphi + \sum_{\substack{j=1\\ \varpi(u_j)=\varpi(\varphi)-|\varpi|}}^{\mu-1} \mathbf{F}\vec{\nabla}u_j. \end{aligned} \tag{3.24}$$

Let $\vec{F} \in Z^2(\mathcal{A}, \varphi)$. According to Remark 3.18, there exists $G, H \in \mathcal{A}$ such that

$$\vec{F} = \vec{\nabla} G + H \vec{\nabla} \varphi. \tag{3.25}$$

Moreover, Proposition 3.16 implies the existence of $\vec{G}_1, \vec{H}_1 \in \mathcal{A}^3$, $N \in \mathbf{N}$ and of elements $\lambda_{i,j}, \delta_{i,j} \in \mathbf{F}$, with $0 \leq i \leq N$ and $0 \leq j \leq \mu - 1$, such that:

$$G = \delta_\varphi^2(\vec{G}_1) + \sum_{i=0}^{N} \sum_{j=0}^{\mu-1} \lambda_{i,j} \varphi^i u_j, \quad H = \delta_\varphi^2(\vec{H}_1) + \sum_{i=0}^{N} \sum_{j=0}^{\mu-1} \delta_{i,j} \varphi^i u_j, \tag{3.26}$$

while we have the 2-coboundaries:

$$\begin{aligned}
&\vec{\nabla}(\delta_\varphi^2(\vec{G}_1)) = -\vec{\nabla}((\vec{\nabla} \times \vec{G}_1) \cdot \vec{\nabla} \varphi) = \delta_\varphi^1(\vec{\nabla} \times \vec{G}_1) \in B^2(\mathcal{A}, \varphi), \\
&\delta_\varphi^2(\vec{H}_1)\, \vec{\nabla} \varphi = -\left((\vec{\nabla} \times \vec{H}_1) \cdot \vec{\nabla} \varphi\right) \vec{\nabla} \varphi = -\delta_\varphi^1(\vec{H}_1 \times \vec{\nabla} \varphi) \in B^2(\mathcal{A}, \varphi).
\end{aligned}$$

Using this fact, (3.25) and (3.26), we obtain

$$\vec{F} \in B^2(\mathcal{A}, \varphi) + \sum_{j=1}^{\mu-1} \operatorname{Cas}(\mathcal{A}, \varphi) \vec{\nabla} u_j + \sum_{j=0}^{\mu-1} \operatorname{Cas}(\mathcal{A}, \varphi) u_j \vec{\nabla} \varphi.$$

Remark 3.21 then implies that $\vec{F}$ can be decomposed as in the right hand side of (3.24). On the other hand, all elements of the right hand side of (3.24) are 2-cocycles, yielding equality in (3.24). (Indeed, using Formula (3.1), we have, for all $F, G \in \mathcal{A}$, $\delta_\varphi^2(\varphi \vec{\nabla} F) = -\vec{\nabla} \varphi \cdot (\vec{\nabla} \times (\varphi \vec{\nabla} F)) = 0$ and $\delta_\varphi^2(G \vec{\nabla} \varphi) = -\vec{\nabla} \varphi \cdot (\vec{\nabla} \times (G \vec{\nabla} \varphi)) = 0$).

Let us show that the sum in (3.24) is direct and let us consider $N \in \mathbf{N}$ and some elements of $\mathbf{F}$: $(\lambda_j, \gamma_{i,l}, \delta_{i,k})$, for $0 \leq i \leq N$, $1 \leq j \leq \mu - 1$ such that $\varpi(u_j) = \varpi(\varphi) - |\varpi|$, $0 \leq k \leq \mu - 1$ such that $\varpi(u_k) = \varpi(\varphi) - |\varpi|$, $1 \leq l \leq \mu - 1$ such that $\varpi(u_l) \neq \varpi(\varphi) - |\varpi|$ and an element $\vec{H} \in \mathcal{A}^3$ satisfying the equation in $\mathfrak{X}^2(\mathcal{A})$:

$$\begin{aligned}
&\sum_{\substack{j=1 \\ \varpi(u_j) = \varpi(\varphi) - |\varpi|}}^{\mu-1} \lambda_j \vec{\nabla} u_j + \sum_{i=0}^{N} \sum_{\substack{k=0 \\ \varpi(u_k) = \varpi(\varphi) - |\varpi|}}^{\mu-1} \delta_{i,k} \varphi^i u_k \vec{\nabla} \varphi \\
&+ \sum_{i=0}^{N} \sum_{\substack{l=1 \\ \varpi(u_l) \neq \varpi(\varphi) - |\varpi|}}^{\mu-1} \gamma_{i,l} \varphi^i \vec{\nabla} u_l = \delta_\varphi^1(\vec{H}) = -\vec{\nabla}(\vec{H} \cdot \vec{\nabla} \varphi) + \operatorname{Div}(\vec{H}) \vec{\nabla} \varphi.
\end{aligned}$$

Observing this equation leads to see that each $\vec{\nabla} u_j$ of the first sum (with $\varpi(u_j) = \varpi(\varphi) - |\varpi|$) is an element of $\mathfrak{X}^2(\mathcal{A})_{\varpi(\varphi) - 2|\varpi|}$, while the other elements in this equation are of strictly bigger degrees. This fact implies $\lambda_j = 0$, for all $1 \leq j \leq \mu - 1$ such that $\varpi(u_j) = \varpi(\varphi) - |\varpi|$. Then, we have:

$$\sum_{i=0}^{N} \sum_{\substack{k=0 \\ \varpi(u_k)=\varpi(\varphi)-|\varpi|}}^{\mu-1} \delta_{i,k}\varphi^i u_k \vec{\nabla}\varphi + \sum_{i=0}^{N} \sum_{\substack{l=1 \\ \varpi(u_l)\neq\varpi(\varphi)-|\varpi|}}^{\mu-1} \gamma_{i,l}\varphi^i \vec{\nabla} u_l$$
$$= -\vec{\nabla}(\vec{H}\cdot\vec{\nabla}\varphi) + \mathrm{Div}(\vec{H})\vec{\nabla}\varphi,$$

that can also be written as:

$$\vec{\nabla}\left(\sum_{i=0}^{N} \sum_{\substack{l=1 \\ \varpi(u_l)\neq\varpi(\varphi)-|\varpi|}}^{\mu-1} \gamma_{i,l}\varphi^i u_l + \vec{H}\cdot\vec{\nabla}\varphi\right) = L\vec{\nabla}\varphi, \tag{3.27}$$

where

$$L := \mathrm{Div}(\vec{H}) - \sum_{i=0}^{N} \sum_{\substack{k=0 \\ \varpi(u_k)=\varpi(\varphi)-|\varpi|}}^{\mu-1} \delta_{i,k}\varphi^i u_k + \sum_{i=1}^{N} \sum_{\substack{l=1 \\ \varpi(u_l)\neq\varpi(\varphi)-|\varpi|}}^{\mu-1} i\gamma_{i,l}\varphi^{i-1} u_l.$$

So the element

$$\sum_{i=0}^{N} \sum_{\substack{l=1 \\ \varpi(u_l)\neq\varpi(\varphi)-|\varpi|}}^{\mu-1} \gamma_{i,l}\varphi^i u_l + \vec{H}\cdot\vec{\nabla}\varphi$$

satisfies the 0-cocycle condition (that is to say, is a Casimir) and Proposition 3.11 implies that there exist elements $c_r \in \mathbf{F}$ with $r \geq 1$, such that:

$$\sum_{i=0}^{N} \sum_{\substack{l=1 \\ \varpi(u_l)\neq\varpi(\varphi)-|\varpi|}}^{\mu-1} \gamma_{i,l}\varphi^i u_l + \vec{H}\cdot\vec{\nabla}\varphi = \sum_{r\geq 1} c_r\varphi^r. \tag{3.28}$$

Therefore, by definition of the u_l, we have $\gamma_{0,l} = 0$, for all l between 1 and $\mu - 1$, satisfying $\varpi(u_l) \neq \varpi(\varphi) - |\varpi|$, so that

$$\vec{H}\cdot\vec{\nabla}\varphi = \sum_{r\geq 1} c_r\varphi^r - \sum_{i=1}^{N} \sum_{\substack{l=1 \\ \varpi(u_l)\neq\varpi(\varphi)-|\varpi|}}^{\mu-1} \gamma_{i,l}\varphi^i u_l$$
$$= \left(\sum_{r\geq 1} \frac{c_r}{\varpi(\varphi)}\varphi^{r-1}\vec{e}_\varpi - \sum_{i=1}^{N} \sum_{\substack{l=1 \\ \varpi(u_l)\neq\varpi(\varphi)-|\varpi|}}^{\mu-1} \frac{\gamma_{i,l}}{\varpi(\varphi)}\varphi^{i-1} u_l\,\vec{e}_\varpi\right)\cdot\vec{\nabla}\varphi.$$

Proposition 3.6 leads to the existence of an element $\vec{K} \in \mathcal{A}^3$ such that:

$$\vec{H} = \sum_{r\geq 1} \frac{c_r}{\varpi(\varphi)}\varphi^{r-1}\vec{e}_\varpi - \sum_{i=1}^{N} \sum_{\substack{l=1 \\ \varpi(u_l)\neq\varpi(\varphi)-|\varpi|}}^{\mu-1} \frac{\gamma_{i,l}}{\varpi(\varphi)}\varphi^{i-1} u_l\,\vec{e}_\varpi + \vec{K}\times\vec{\nabla}\varphi.$$

Using Formulas (3.6) and (3.3),

$$\begin{aligned}\operatorname{Div}(\vec{H}) &= \sum_{r\geq 1}\frac{c_r}{\varpi(\varphi)}(\varpi(\varphi)(r-1)+|\varpi|)\varphi^{r-1}\\ &-\sum_{i=1}^{N}\sum_{\substack{l=1\\ \varpi(u_l)\neq\varpi(\varphi)-|\varpi|}}^{\mu-1}\frac{\gamma_{i,l}}{\varpi(\varphi)}(\varpi(\varphi)(i-1)+\varpi(u_l)+|\varpi|)\varphi^{i-1}u_l\\ &+(\vec{\nabla}\times\vec{K})\cdot\vec{\nabla}\varphi. \end{aligned}\tag{3.29}$$

Moreover, Equations (3.27) and (3.28) give $L=\sum_{r\geq 1} rc_r\varphi^{r-1}$, that implies, with the expression of L:

$$\begin{aligned}\operatorname{Div}(\vec{H}) = \sum_{r\geq 1} rc_r\varphi^{r-1} &+ \sum_{i=0}^{N}\sum_{\substack{k=0\\ \varpi(u_k)=\varpi(\varphi)-|\varpi|}}^{\mu-1}\delta_{i,k}\varphi^i u_k\\ &-\sum_{i=1}^{N}\sum_{\substack{l=1\\ \varpi(u_l)\neq\varpi(\varphi)-|\varpi|}}^{\mu-1} i\gamma_{i,l}\varphi^{i-1}u_l. \end{aligned}\tag{3.30}$$

Now, using the two expressions of $\operatorname{Div}(\vec{H})$ in (3.29) and (3.30), we obtain:

$$\begin{aligned}\delta^2_\varphi(\vec{K}) = (\vec{\nabla}\times\vec{K})\cdot\vec{\nabla}\varphi &= \sum_{r\geq 1}\frac{c_r}{\varpi(\varphi)}(\varpi(\varphi)-|\varpi|)\varphi^{r-1}\\ +\sum_{i=1}^{N}\sum_{\substack{l=1\\ \varpi(u_l)\neq\varpi(\varphi)-|\varpi|}}^{\mu-1}&\frac{\gamma_{i,l}}{\varpi(\varphi)}(\varpi(u_l)-\varpi(\varphi)+|\varpi|)\varphi^{i-1}u_l\\ +\sum_{i=0}^{N}\sum_{\substack{k=0\\ \varpi(u_k)=\varpi(\varphi)-|\varpi|}}^{\mu-1}&\delta_{i,k}\varphi^i u_k \in \bigoplus_{j=0}^{\mu-1}\operatorname{Cas}(\mathcal{A},\varphi)\,u_j. \end{aligned}$$

Proposition 3.16 implies that, for all $\varpi(\varphi)$ (equal or not to $|\varpi|$), we have $\gamma_{i,l}=0$ and $\delta_{i,k}=0$, for all i between 0 and N, all l between 1 and $\mu-1$, with $\varpi(u_l)\neq\varpi(\varphi)-|\varpi|$ and all k between 0 and $\mu-1$ satisfying $\varpi(u_k)=\varpi(\varphi)-|\varpi|$.

Therefore all the elements λ_j, $\gamma_{i,l}$ and $\delta_{i,k}$ considered are equal to zero, hence the fact that the sum is direct. We have so obtained the result desired for $H^2(\mathcal{A},\varphi)$. $\square$

Remark 3.22. Using Euler's Formula (3.5) and the writings of the Poisson cohomology spaces $H^1(\mathcal{A},\varphi)$ and $H^2(\mathcal{A},\varphi)$ given in Propositions 3.14 and 3.19, we can make the ring structure on the space $H^\bullet(\mathcal{A},\varphi):=\bigoplus_{k=0}^{3}H^k(\mathcal{A},\varphi)$, induced by the wedge product, explicit. One obtains, for example, that

$$\wedge: H^1(\mathcal{A},\varphi)\times H^2(\mathcal{A},\varphi)\longrightarrow H^3(\mathcal{A},\varphi)$$

is surjective when $\varpi(\varphi)=|\varpi|$.

3.3 Homology of $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$

In this section, we consider the affine space $\mathbf{F}^3$ and its algebra of regular functions $\mathcal{A} = \mathbf{F}[x, y, z]$ (with $\mathrm{char}(\mathbf{F}) = 0$) and $\varphi \in \mathcal{A}$, a weight homogeneous polynomial with an isolated singularity. This algebra is still equipped with the Poisson structure $\{\cdot\,,\cdot\}_\varphi$. We use the Poisson cohomology of the Poisson variety $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$, given in the last section 3.2, to determine its Poisson homology. See Paragraph 2.2.2, for the definition of the Poisson homology.

The boundary operator of the algebra $(\mathcal{A}, \{\cdot\,,\cdot\}_\varphi)$ is denoted by δ_k^φ, while the Poisson homology spaces are denoted by $H_k(\mathcal{A}, \varphi)$. In the particular case of our polynomial algebra $\mathcal{A} = \mathbf{F}[x, y, z]$, we recall (See Paragraph 3.1.2, for details) that $\Omega^\bullet(\mathcal{A})$ is the $\mathcal{A}$-module generated by the wedge products of the 1-differential forms $\mathsf{d}x, \mathsf{d}y, \mathsf{d}z$ and that we have $\Omega^i(\mathcal{A}) = \{0\}$, for all $i \geq 4$ and the isomorphisms (given by the operator $*$):

$$\Omega^0(\mathcal{A}) \simeq \Omega^3(\mathcal{A}) \simeq \mathcal{A}, \qquad \Omega^1(\mathcal{A}) \simeq \Omega^2(\mathcal{A}) \simeq \mathcal{A}^3, \tag{3.31}$$

which allow us to use the same notations and formulas than in the last section, when we talk about differential forms. For example, the 1-differential form $\mathsf{d}\varphi$ corresponds, with these notations, to the element $\vec{\nabla}\varphi$ of $\mathcal{A}^3$, as the biderivation $\{\cdot\,,\cdot\}_\varphi$. In terms of the operator $*$, we have indeed $*(\{\cdot\,,\cdot\}_\varphi) = \mathsf{d}\varphi$.

Proposition 3.23. *If $\varphi \in \mathcal{A}$ is weight homogeneous with an isolated singularity, the homology spaces of $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$ are given by:*

$$H_k(\mathcal{A}, \varphi) \simeq H^{3-k}(\mathcal{A}, \varphi), \text{ for all } k = 0, 1, 2, 3. \tag{3.32}$$

Proof. This result easily comes from the isomorphisms between the spaces $\Omega^k(\mathcal{A})$ and $\mathfrak{X}^{3-k}(\mathcal{A})$, given in Paragraph 3.1.2, and from the fact that, under these identifications, we get

$$\partial_k^\varphi = (-1)^k\, \delta_\varphi^{3-k}. \tag{3.33}$$

□

Remark 3.24 (Modular derivation). Another way to see the isomorphisms (3.32) is to observe that the modular derivation (See section 2.2.3) of the Poisson algebra $(\mathcal{A}, \{\cdot\,,\cdot\}_\varphi)$, given by $\mathrm{Div}(\{\cdot\,,\cdot\}_\varphi)$, is equal to zero, as, by definition:

$$*(\mathrm{Div}(\{\cdot\,,\cdot\}_\varphi)) = \mathsf{d} * (\{\cdot\,,\cdot\}_\varphi) = \mathsf{d}(\mathsf{d}\varphi) = 0.$$

Thus $(\mathcal{A}, \{\cdot\,,\cdot\}_\varphi)$ is unimodular and, according to Proposition 2.28, in this case, there exists a duality (expressed by the isomorphisms (3.32)) between the Poisson cohomology and Poisson homology spaces of the Poisson algebra $(\mathcal{A}, \{\cdot\,,\cdot\}_\varphi)$.

4

Poisson cohomology and homology for surfaces in $\mathbf{F}^3$

In this chapter, we rather consider affine varieties that are surfaces in $\mathbf{F}^3$. In our study, one finds two types of such surfaces: a smooth one, the affine space $\mathbf{F}^2$; and a singular one, $\mathcal{F}_\varphi : \{\varphi = 0\}$, where $\varphi \in \mathbf{F}[x, y, z]$ is a weight homogeneous polynomial with an isolated singularity. We have already seen in Paragraphes 2.1.2 and 2.1.3 that we can equip both these affine surfaces with Poisson structures. The space $\mathbf{F}^2$ will be endowed with a Poisson structure that admits a singular locus, while the singular surface $\mathcal{F}_\varphi$ will be endowed with a Poisson structure, as regular as possible, that is to say, symplectic everywhere, except on the (isolated) singularity. We recall these constructions and the corresponding Poisson (co)homology complexes and determine the Poisson cohomology and homology of the affine Poisson surfaces obtained.

4.1 Multi-Derivations and Kähler differentials of $\mathbf{F}[x, y]$

In this section, we will study the particular case of the skew-symmetric multi-derivations and Kähler differentials of the polynomial algebra $\mathcal{A} := \mathbf{F}[x, y]$. It will be very close to the considerations we have done in dimension three. In fact, we will do some identifications of the spaces of all skew-symmetric k-derivations and all Kähler differentials with $\mathcal{A}$ or $\mathcal{A}^2$, that will be useful for the further computations.

4.1.1 Multi-derivations

Let us consider the affine variety $\mathbf{F}^2$, equipped with its algebra of regular functions $\mathcal{A} = \mathbf{F}[x, y]$. We will study the skew-symmetric multi-derivations of this polynomial algebra.

By definition, we have $\mathfrak{X}^0(\mathcal{A}) = \mathcal{A}$ and according to Remark 2.14, we have $\mathfrak{X}^k(\mathcal{A}) \simeq \{0\}$, for $k \geq 3$. It remains to study the two spaces $\mathfrak{X}^1(\mathcal{A})$ and $\mathfrak{X}^2(\mathcal{A})$. Let us consider the derivations of $\mathcal{A}$. According to Proposition 2.15, such an element $\mathcal{V} \in \mathfrak{X}^1(\mathcal{A})$ is completely defined by its values on the two generators x

and y. If we denote by $F \in \mathcal{A}$, the element $F = \mathcal{V}[x]$ and by $G \in \mathcal{A}$, the element $G = \mathcal{V}[y]$, then we have indeed :

$$\mathcal{V} = F\,\frac{\partial}{\partial x} + G\,\frac{\partial}{\partial y}.$$

So we have a one-to-one correspondence between $\mathfrak{X}^1(\mathcal{A})$ and $\mathcal{A}^2$, given by:

$$\begin{array}{ccc} \mathfrak{X}^1(\mathcal{A}) & \longleftrightarrow & \mathcal{A}^2 \\ \mathcal{V} & \longrightarrow & (\mathcal{V}[x], \mathcal{V}[y]) \\ F\,\frac{\partial}{\partial x} + G\,\frac{\partial}{\partial y} & \longleftarrow & (F, G) \end{array}$$

Now, let us turn out with the skew-symmetric biderivations of $\mathcal{A}$. As previously, an element $\mathcal{W} \in \mathfrak{X}^2(\mathcal{A})$ is defined by its values on the two generators x and y. Let us consider the element $H \in \mathcal{A}$ defined by: $H = \mathcal{W}[x, y]$, then we have:

$$\mathcal{W} = H\,\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

One more time, we have a correspondence between $\mathfrak{X}^2(\mathcal{A})$ and $\mathcal{A}$:

$$\begin{array}{ccc} \mathfrak{X}^2(\mathcal{A}) & \longleftrightarrow & \mathcal{A} \\ \mathcal{W} & \longrightarrow & \mathcal{W}[x, y] \\ H\,\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} & \longleftarrow & H \end{array}$$

Vector Formulas

According to the identifications we have considered in the previous paragraph, we will often write the skew-symmetric multi-derivations of $\mathcal{A}$ as elements of $\mathcal{A}$ or $\mathcal{A}^2$. In order to simplify the writing of the next computations and in a very analogous way than for the polynomial algebra $\mathbf{F}[x, y, z]$, we will, in this paragraph, introduce some notations in $\mathcal{A}$ and $\mathcal{A}^2$. The elements of $\mathcal{A}^2$ will often be denoted with an arrow, like $\vec{F} = (F_1, F_2) \in \mathcal{A}^2$.

Let $\vec{F}, \vec{G} \in \mathcal{A}^2$ be two elements of $\mathcal{A}^2$, with $\vec{F} = (F_1, F_2)$ and $\vec{G} = (G_1, G_2)$ and let us consider two polynomials $H, K \in \mathcal{A}$.

In $\mathcal{A}^2$, the usual inner product is denoted by $\cdot$, so that:

$$\vec{F} \cdot \vec{G} = F_1 G_1 + F_2 G_2 \in \mathcal{A}.$$

We now denote by $\vec{\nabla}$ the gradient operator:

$$\vec{\nabla} H = \left(\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}\right) \in \mathcal{A}^2,$$

and we define another operator $\vec{\mathcal{H}} : \mathcal{A} \to \mathcal{A}^2 : K \mapsto \vec{\mathcal{H}}_K$, with the formula:

$$\vec{\mathcal{H}}_K = \left(\frac{\partial K}{\partial y}, -\frac{\partial K}{\partial x}\right) \in \mathcal{A}^2.$$

Finally, the classical divergence operator is given by:

$$\mathrm{Div}(\vec{F}\,) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \in \mathcal{A}.$$

In the computations in $\mathcal{A}$ and $\mathcal{A}^2$ that will appear in next sections, we will need the following formulas, easily verified $\vec{G} \in \mathcal{A}^2$ and $H, K \in \mathcal{A}$:

$$\mathrm{Div}(H\vec{G}) = \vec{\nabla} H \cdot \vec{G} + H\,\mathrm{Div}(\vec{G}), \tag{4.1}$$

$$\mathrm{Div}(\vec{\mathcal{H}}_K) = 0, \tag{4.2}$$

$$\vec{\nabla} H \cdot \vec{\mathcal{H}}_K = -\vec{\nabla} K \cdot \vec{\mathcal{H}}_H. \tag{4.3}$$

Weight homogeneity in dimension two

Assume that we are in a weight homogeneous context, i.e., assume that we have some fixed weights, ϖ_1 and ϖ_2, for the two variables x and y. Then, the (weighted) Euler derivation (2.35) is, in the case of dimension two, given by $\vec{e}_\varpi = \varpi_1\, x\frac{\partial}{\partial x} + \varpi_2\, y\frac{\partial}{\partial y}$ and, under the previous identifications, it can be viewed as the element (also denoted by $\vec{e}_\varpi$)

$$\vec{e}_\varpi = (\varpi_1\, x, \varpi_2\, y) \in \mathcal{A}^2.$$

We then have $\mathrm{Div}(\vec{e}_\varpi) = \varpi_1 + \varpi_2$. With the help of the notations of the previous paragraph, Euler's formula (2.36) for a weight homogeneous polynomial $G \in \mathcal{A}$ can also be written as:

$$\vec{\nabla} G \cdot \vec{e}_\varpi = \varpi(G) G. \tag{4.4}$$

Using this formula and (4.1), we obtain the following

$$\mathrm{Div}(G\vec{e}_\varpi) = (\varpi(G) + \varpi_1 + \varpi_2) G. \tag{4.5}$$

Finally, using the notations of paragraph 2.3.1, the previous identifications of the multi-derivations of $\mathcal{A}$ and the remark 2.31, we have the following isomorphisms:

$$\begin{aligned} \mathfrak{X}^0(\mathcal{A})_i &\simeq \mathcal{A}_i, \\ \mathfrak{X}^1(\mathcal{A})_i &\simeq \mathcal{A}_{i+\varpi_1} \times \mathcal{A}_{i+\varpi_2}, \\ \mathfrak{X}^2(\mathcal{A})_i &\simeq \mathcal{A}_{i+\varpi_1+\varpi_2}. \end{aligned} \tag{4.6}$$

These decompositions show us that the previous identifications of the spaces of all multi-derivations of $\mathcal{A}$ (in the last paragraph) do not respect the weight.

4.1.2 Kähler differentials

Let us turn out with the Kähler differentials of the algebra $\mathcal{A} = \mathbf{F}[x, y]$. We have $\Omega^0(\mathcal{A}) = \mathcal{A}$ and $\Omega^k(\mathcal{A}) \simeq \{0\}$, for all $k \geq 3$. Now, let $\alpha \in \Omega^1(\mathcal{A})$ be a Kähler 1-differential of $\mathcal{A}$. In view of the definition of the Kähler differentials, there exist some polynomials $F, G \in \mathcal{A}$ such that:

$$\alpha = F\,\mathsf{d}x + G\,\mathsf{d}y,$$

and we have the following correspondence between $\Omega^1(\mathcal{A})$ and $\mathcal{A}^2$:

$$\begin{aligned} \Omega^1(\mathcal{A}) \quad &\longleftrightarrow \quad \mathcal{A}^2 \\ F\,\mathsf{d}x + G\,\mathsf{d}y &\longleftrightarrow (F, G) \end{aligned}$$

Let $\beta \in \Omega^2(\mathcal{A})$ be a Kähler 2-differential of $\mathcal{A}$, then, according to the definition of $\Omega^\bullet(\mathcal{A})$, there exists $H \in \mathcal{A}$ satisfying:

$$\beta = H\,\mathsf{d}x \wedge \mathsf{d}y,$$

that leads to the correspondence between $\Omega^2(\mathcal{A})$ and $\mathcal{A}$:

$$\begin{aligned} \Omega^2(\mathcal{A}) \quad &\longleftrightarrow \mathcal{A} \\ H\,\mathsf{d}x \wedge \mathsf{d}y &\longleftrightarrow H \end{aligned}$$

Summarizing, we have the following natural isomorphisms (see the paragraph 4.1.1 for the skew-symmetric multi-derivations of $\mathcal{A} = \mathbf{F}[x, y]$):

$$\begin{aligned} \Omega^0(\mathcal{A}) &\simeq \mathcal{A} \simeq \mathfrak{X}^2(\mathcal{A}), \\ \Omega^1(\mathcal{A}) &\simeq \mathcal{A}^2 \simeq \mathfrak{X}^1(\mathcal{A}), \\ \Omega^2(\mathcal{A}) &\simeq \mathcal{A} \simeq \mathfrak{X}^0(\mathcal{A}). \end{aligned} \tag{4.7}$$

We recall that these natural isomorphisms are given by the star operator $\star$, defined in the paragraph 2.2.2 with the volume form $\lambda = \mathsf{d}x \wedge \mathsf{d}y$. We have, for $F, G \in \mathcal{A}$,

$$\begin{aligned} \star(F) &= \imath_F \lambda = F\mathsf{d}x \wedge \mathsf{d}y, \\ \star\Big(F\tfrac{\partial}{\partial x} + G\tfrac{\partial}{\partial y}\Big) &= \imath_{F\frac{\partial}{\partial x} + G\frac{\partial}{\partial y}} \lambda = F\mathsf{d}y - G\mathsf{d}x, \\ \star\Big(F\tfrac{\partial}{\partial x} \wedge \tfrac{\partial}{\partial y}\Big) &= \imath_{F\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}} \lambda = F, \end{aligned} \tag{4.8}$$

so that, explicitly, the above isomorphisms are given by:

$$
\begin{array}{ccc}
\Omega^0(\mathcal{A}) & \stackrel{*}{\longleftrightarrow} & \mathfrak{X}^2(\mathcal{A}) \\
H & \longleftrightarrow & H\,\dfrac{\partial}{\partial x}\wedge\dfrac{\partial}{\partial y}
\end{array}
$$

$$
\begin{array}{ccc}
\Omega^1(\mathcal{A}) & \stackrel{*}{\longleftrightarrow} & \mathfrak{X}^1(\mathcal{A}) \\
F\,\mathsf{d}x + G\,\mathsf{d}y & \longleftrightarrow & G\,\dfrac{\partial}{\partial x} - F\,\dfrac{\partial}{\partial y}
\end{array}
$$

$$
\begin{array}{ccc}
\Omega^2(\mathcal{A}) & \stackrel{*}{\longleftrightarrow} & \mathfrak{X}^0(\mathcal{A}) \\
F\,\mathsf{d}x\wedge\mathsf{d}y & \longleftrightarrow & F
\end{array}
$$

We point out that the isomorphism between $\Omega^1(\mathcal{A})$ and $\mathfrak{X}^1(\mathcal{A})$ is, in terms of elements of $\mathcal{A}^2$, not the identity but given by:

$$
\begin{aligned}
\mathcal{A}^2 \simeq \Omega^1(\mathcal{A}) &\longleftrightarrow \mathfrak{X}^1(\mathcal{A}) \simeq \mathcal{A}^2 \\
(F,G) &\longrightarrow (G,-F) \\
(-G,F) &\longleftarrow (F,G)
\end{aligned}
\tag{4.9}
$$

We will now consider the de Rham complex in the particular case of the algebra $\mathcal{A}=\mathbf{F}[x,y]$. According to the previous identifications, we will be able to write it in terms of elements of $\mathcal{A}$ and $\mathcal{A}^2$.

For $F\in\mathcal{A}=\Omega^0(\mathcal{A})$, we have,

$$
\mathsf{d}F = \frac{\partial F}{\partial x}\,\mathsf{d}x + \frac{\partial F}{\partial y}\,\mathsf{d}y \;\in \Omega^1(\mathcal{A}),
$$

which corresponds, under the isomorphisms (4.7), to $\Big(\dfrac{\partial F}{\partial x},\dfrac{\partial F}{\partial y}\Big)\in\mathcal{A}^2$.

Now, for $\alpha = F\,\mathsf{d}x + G\,\mathsf{d}y \in \Omega^1(\mathcal{A})$, with $F,G\in\mathcal{A}$, the de Rham differential leads to the element

$$
\mathsf{d}\alpha = \Big(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\Big)\mathsf{d}x\wedge\mathsf{d}y \;\in \Omega^2(\mathcal{A}),
$$

that one can see, under the isomorphisms (4.7), as $\dfrac{\partial G}{\partial x} - \dfrac{\partial F}{\partial y}\in\mathcal{A}$.

Finally, if $\beta\in\Omega^2(\mathcal{A})$ is a Kähler 2-differential, then $\mathsf{d}\beta = 0$.

So that, the de Rham complex of the algebra $\mathcal{A}=\mathbf{F}[x,y]$, in terms of elements of $\mathcal{A}$ and $\mathcal{A}^2$, can be written as:

$$
\begin{array}{ccccccc}
\mathbf{F} \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{A}^2 & \longrightarrow & \mathcal{A} \\
& H & \longmapsto & \Big(\dfrac{\partial H}{\partial x},\dfrac{\partial H}{\partial y}\Big) & & \\
& & & (F,G) & \longmapsto & \dfrac{\partial G}{\partial x} - \dfrac{\partial F}{\partial y}
\end{array}
\tag{4.10}
$$

Proposition 4.1. *The algebraic de Rham complex of the polynomial algebra* $\mathcal{A} = \mathbf{F}[x,y]$ *is an exact one.*

Proof. We will work with the algebraic de Rham complex of $\mathcal{A}$, written as in (4.10), in terms of $\mathcal{A}$ and $\mathcal{A}^2$. As in dimension three, the classical argument of exactness of the de Rham complex of $C^\infty(\mathbf{R}^n)$ is easily adapted to the algebraic case.

The only fact to verify is: if $(F,G) \in \mathcal{A}^2$ satisfies $\partial G/\partial x - \partial F/\partial y = 0$, then there exists $H \in \mathcal{A}$ such that $(F,G) = \Big(\partial H/\partial x\,,\, \partial H/\partial y\Big)$. So let $(F,G) \in \mathcal{A}^2$ be two polynomials of $\mathcal{A} = \mathbf{F}[x,y]$. Assume that they are homogeneous of the same degree $r \in \mathbf{N}$. Then Euler's Formula (2.36) for homogeneous polynomials (that is to say for $\varpi_1 = \varpi_2 = \varpi_3 = 1$) says:

$$r\,F = x\,\frac{\partial F}{\partial x} + y\,\frac{\partial F}{\partial y} \quad \text{and} \quad r\,G = x\,\frac{\partial G}{\partial x} + y\,\frac{\partial G}{\partial y}.$$

Now, suppose that $\dfrac{\partial G}{\partial x} = \dfrac{\partial F}{\partial y}$ and let $H \in \mathcal{A}$ be the polynomial $H = \dfrac{1}{r+1}\Big(x\,F + y\,G\Big)$, then, we obtain:

$$\frac{\partial H}{\partial x} = \frac{1}{r+1}\Big(F + x\,\frac{\partial F}{\partial x} + y\,\frac{\partial G}{\partial x}\Big) = \frac{1}{r+1}\Big(F + x\,\frac{\partial F}{\partial x} + y\,\frac{\partial F}{\partial y}\Big) = F,$$

and, similarly, $\partial H/\partial y = G$, so that $(F,G) = \left(\dfrac{\partial H}{\partial x}, \dfrac{\partial H}{\partial y}\right)$ and the algebraic de Rham complex of $\mathcal{A}$ is exact. □

As, in the further chapters, we will rather work with multi-derivations than Kähler differentials, we will here express the de Rham complex of $\mathcal{A} = \mathbf{F}[x,y]$, in terms of elements of $\mathcal{A}$ and $\mathcal{A}^2$, but viewed as multi-derivations. For this purpose, we translate the complex (4.10) with the help of the correspondence (4.9). It then becomes

$$\begin{array}{ccccccc} \mathbf{F} & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{A}^2 & \longrightarrow & \mathcal{A} \\ & & K & \longmapsto & \vec{\mathcal{H}}_K & & \\ & & & & \vec{H} & \longmapsto & \operatorname{Div}(\vec{H}\,) \end{array} \tag{4.11}$$

Remark 4.2. This complex is an other writing of the de Rham complex, so that, Proposition 4.1 implies that the complex (4.11) is also an exact one: if $\vec{H} \in \mathcal{A}^2$ satisfies $\operatorname{Div}(\vec{H}\,) = 0$, then there exists $K \in \mathcal{A}$ such that $\vec{H} = \vec{\mathcal{H}}_K$. (By adapting the above proof, we see that, if $\vec{H} = (F,G)$ with F and G two homogeneous polynomials of the same degree $r \in \mathbf{N}$, then we can choose K to be the element $K = \frac{1}{r+1}(-xG + yF)$.)

4.1.3 Isolated singularities and dimension two

In this paragraph, we want to show the link between isolated singularities and square-free property, for polynomials in $\mathbf{F}[x,y]$.

Lemma 4.3. *Let $\psi \in \mathbf{F}[x,y]$ be a weight homogeneous polynomial. The corresponding weights of x and y are denoted by ϖ_1 and ϖ_2 and we suppose that $\varpi(\psi) > \varpi_i$, for $i = 1,2$, so that ψ has a singularity at the origin. Then, the following conditions are equivalent:*

(i) ψ has an isolated singularity (at the origin), i.e., the $\mathbf{F}$-vector space $\mathcal{A}_{sing}(\psi) = \dfrac{\mathbf{F}[x,y]}{\langle \frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}\rangle}$ is of finite dimension, denoted by μ_ψ (see Paragraph 2.3.2);
(ii) ψ is square-free (i.e., any polynomial $\xi \in \mathbf{F}[x,y]$, such that ξ^2 divides ψ in $\mathbf{F}[x,y]$, is necessarily a constante);
(iii) The first order derivatives of ψ, $\frac{\partial\psi}{\partial x}$ and $\frac{\partial\psi}{\partial y}$, are coprime.

Proof. First, let us suppose that ψ is not square free: there exists a non constant polynomial $\xi \in \mathbf{F}[x,y]$ and $\chi \in \mathbf{F}[x,y]$, satisfying $\psi = \xi^2\chi$. Then, we have:

$$\frac{\partial\psi}{\partial x} = 2\xi\chi\frac{\partial\xi}{\partial x} + \xi^2\frac{\partial\chi}{\partial x}, \qquad \frac{\partial\psi}{\partial y} = 2\xi\chi\frac{\partial\xi}{\partial y} + \xi^2\frac{\partial\chi}{\partial y}, \tag{4.12}$$

so that ξ divides the first order partial derivatives of ψ. Any polynomial of $\mathbf{F}[x,y]$, no multiple of ξ, has then a non trivial projection in the quotient algebra $\mathcal{A}_{sing}(\psi)$ and it implies that this $\mathbf{F}$-vector space is of infinite dimension. This shows that, if ψ has an isolated singularity, then ψ is square-free ((*i*) implies (*ii*)).

Now, let us show that ψ is square free, if and only if, the first order partial derivatives of ψ have no (non constant) common factor. Indeed, if there exists $\chi \in \mathbf{F}[x,y]$, dividing each partial derivatives of ψ, according to Euler's Formula (2.36), χ divides ψ. We write now $\psi = \chi\varphi$, we suppose χ irreducible and we have,

$$\frac{\partial\psi}{\partial x} = \frac{\partial(\chi\varphi)}{\partial x} = \chi\frac{\partial\varphi}{\partial x} + \varphi\frac{\partial\chi}{\partial x}, \qquad \frac{\partial\psi}{\partial y} = \frac{\partial(\chi\varphi)}{\partial y} = \chi\frac{\partial\varphi}{\partial y} + \varphi\frac{\partial\chi}{\partial y},$$

so that χ divides φ, and χ^2 divides ψ. Under the hypothesis done on ψ, χ is constant and the partial derivatives of ψ are coprime. Conversely, if ψ is not square-free, $\psi = \xi^2\chi$ and (4.12) implies that $\frac{\partial\psi}{\partial x}$ and $\frac{\partial\psi}{\partial y}$ are not coprime. So that, we have obtained that conditions (*ii*) and (*iii*) are equivalent.

We now are supposing that ψ is square free, or, equivalently, that its first order partial derivatives are coprime. We show that, this hypothesis implies that the origin is an isolated singularity for ψ. We refer to [49], for the proofs of the following lemma and theorem.

Lemma 4.4 (from [49]). *Let $F, G \in \mathbf{F}[x,y]$ be two non zero coprime polynomials. Then there exists a non zero polynomial $D \in \mathbf{F}[x]$ in the ideal $\langle F, G\rangle$.*

Proof. Two polynomials F, G which are coprime in $\mathbf{F}[x,y]$ are also coprime in $\mathbf{F}(x)[y]$, so that, according to the Bezout theorem, there exist two polynomials $U, V \in \mathbf{F}(x)[y]$, such that:

$$1 = U(x,y)F + V(x,y)G.$$

We denote by $D(x) \in \mathbf{F}[x]$ the ppcm of the denominators (elements of $\mathbf{F}[x]$) of the rational functions that are the coefficients of U, V, as elements of $\mathbf{F}(x)[y]$. Then, we have:

$$D(x) = AF + BG,$$

where $A = D(x)U \in \mathbf{F}[x,y]$ and $B = D(x)V \in \mathbf{F}[x,y]$, that proves the lemma. □

This lemma allows us to prove the following theorem.

Theorem 4.5 (from [49]). *Let $F, G \in \mathbf{F}[x,y]$ be two non zero coprime polynomials. Then the $\mathbf{F}$-vector space $\mathbf{F}[x,y]/\langle F, G\rangle$ is of finite-dimension.*

Proof. According to the above lemma, there exists a non zero polynomial $D \in \mathbf{F}[x]$ and two polynomials $A, B \in \mathbf{F}[x,y]$ satisfying:

$$D(x) = AF + BG.$$

Let us denote by d the degree of D. Let $i, j \in \mathbf{N}$, then, if $i > d$, we have:

$$x^i y^j \in \langle D\rangle + \langle x^{i-1}y^j, x^{i-2}y^j, \ldots, y^j\rangle \subset \langle F, G\rangle + \langle x^{i-1}y^j, x^{i-2}y^j, \ldots, y^j\rangle,$$

so that, for fixed $j \in \mathbf{N}$, there are only a finite number of $i \in \mathbf{N}$ such that the family of monomials $(x^i y^j)_{i\in\mathbf{N}}$ is free in the quotient vector field $\mathbf{F}[x,y]/\langle F, G\rangle$. If we do the same reasoning for j, we obtain that there is only a finite number of monomials $x^i y^j$ that freely generate $\mathbf{F}[x,y]/\langle F, G\rangle$. That leads to the finite-dimension of this $\mathbf{F}$-vector space. □

Now, according to the above theorem, specialized to $F = \dfrac{\partial\psi}{\partial x}$, $G = \dfrac{\partial\psi}{\partial y}$, we have that, if ψ is square free, or, equivalently, if its first order partial derivatives are coprime, then the origin is an isolated singularity for ψ. □

The Koszul complex in dimension two

Let $\psi \in \mathcal{A} = \mathbf{F}[x,y]$ be a square free weight homogeneous polynomial. Then, we will show that the Koszul complex associated to ψ is an exact one. In fact, according to Lemma 4.3 and the Cohen-Macaulay theorem 3.4, we have that $\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}$ is a regular sequence, so that the Koszul complex associated to ψ is exact. But we use the previous paragraph to obtain the fact that $\frac{\partial\psi}{\partial x}$ and $\frac{\partial\psi}{\partial y}$ are coprime and this permits us to write directly the exactness of the Koszul complex. In fact, according to the identifications of the paragraph 4.1.2, we can rewrite the Koszul complex, in terms of elements of $\mathcal{A}$ and $\mathcal{A}^2$, as follows:

$$\{0\} \longrightarrow \mathcal{A} \xrightarrow{\vec{\nabla}\psi} \mathcal{A}^2 \xrightarrow{\cdot\vec{\mathcal{H}}_\psi} \mathcal{A} \tag{4.13}$$

As for the de Rham complex, we can write the Koszul complex, in terms of elements of $\mathcal{A}$ and $\mathcal{A}^2$, but, viewed as multi-derivations, instead of Kähler differentials. Using (4.9), we obtain the following complex (totally equivalent to (4.13)):

$$\{0\} \longrightarrow \mathcal{A} \xrightarrow{\vec{\mathcal{H}}_\psi} \mathcal{A}^2 \xrightarrow{\cdot(-\vec{\nabla}\psi)} \mathcal{A} \tag{4.14}$$

Proposition 4.6. *If $\psi \in \mathcal{A}$ is a square free weight homogeneous polynomial, then the Koszul complex associated to ψ (4.13) (equivalent to (4.14)) is exact.*

Proof. In order to show that this complex is exact, we will, for example, use the writing (4.13). Let us consider an element $\vec{F} = (F, G) \in \mathcal{A}^2$, satisfying $\vec{F}\cdot\vec{\mathcal{H}}_\psi = 0$. We have to verify that there exists an element $H \in \mathcal{A}$, such that $\vec{F} = H\vec{\nabla}\psi$. The condition $\vec{F}\cdot\vec{\mathcal{H}}_\psi = 0$ can be written as follows:

$$F\frac{\partial\psi}{\partial y} = G\frac{\partial\psi}{\partial x}.$$

As ψ is square free, we know (according to Lemma 4.3), that the first order derivatives of ψ are coprime, so that, necessarily, $\frac{\partial\psi}{\partial x}$ divides F: $F = \frac{\partial\psi}{\partial x}H$, with $H \in \mathcal{A}$. Thus, we also obtain $G = \frac{\partial\psi}{\partial y}H$, and $\vec{F} = H\vec{\nabla}\psi$ and the Koszul complex associated to ψ is exact. □

4.2 Cohomology and homology of $\mathbf{F}^2$

In this section, we will determine the Poisson cohomology of the affine space of dimension two, equipped with a weight homogeneous Poisson structure. In fact, explicit basis of the Poisson cohomology **C**-vector spaces have been determined by P. Monnier, in his thesis and in [45], in an germified (local) context, while the dimensions of the Poisson cohomology **F**-vector spaces have been obtained by P. Vanhaecke and C. Roger, in an algebraic homogeneous context, see [54]. Here, we give explicit basis of the Poisson cohomology spaces, in an algebraic weight homogeneous context. This work is inspired from the works cited above.

4.2.1 Poisson cohomology of $(\mathbf{F}^2, \{\cdot\,,\cdot\}^\psi)$

We consider the affine space of dimension two, $\mathbf{F}^2$, equipped with its algebra of polynomial functions $\mathcal{A} = \mathbf{F}[x, y]$ and with a Poisson structure, defined by a polynomial $\psi \in \mathcal{A}$:

$$\{\cdot\,,\cdot\}^{\psi} := \psi\,\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}.$$

According to the paragraph 4.1.1, we can write the Poisson cohomology complex of $(\mathcal{A},\{\cdot\,,\cdot\}^{\psi})$, in terms of elements of $\mathcal{A}$ and $\mathcal{A}^2$. For example, let $H\in\mathcal{A}\simeq\mathfrak{X}^0(\mathcal{A})$ be a polynomial. Then, the Poisson coboundary operator associated to $(\mathbf{F}^2,\{\cdot\,,\cdot\}^{\psi})$ is denoted by δ^k. Applied on H, it is completely defined by its values on x and y:

$$\delta^0(H)[x] = \{x,H\}^{\psi} = \psi\frac{\partial H}{\partial y},\qquad \delta^0(H)[y] = \{y,H\}^{\psi} = -\psi\frac{\partial H}{\partial x},$$

so that, viewed as an element of $\mathcal{A}^2$, $\delta^0(H)$ is equal to $\left(\psi\,\frac{\partial H}{\partial y},-\psi\,\frac{\partial H}{\partial x}\right)=\psi\vec{\mathcal{H}}_H$ (see paragraph 4.1.3 for this notation). Now, let $\mathcal{V}=F\frac{\partial}{\partial x}+G\frac{\partial}{\partial y}\in\mathfrak{X}^1(\mathcal{A})$ (with $F,G\in\mathcal{A}$) be a derivation of $\mathcal{A}$. Using the identifications of the paragraph 4.1.1, the derivation $\mathcal{V}$ is also viewed as the element $\vec{F}=(F,G)\in\mathcal{A}^2$. We have:

$$\begin{aligned}\delta^1(\mathcal{V})[x,y] &= \{x,\mathcal{V}[y]\}^{\psi}-\{y,\mathcal{V}[x]\}^{\psi}-\mathcal{V}[\{x,y\}^{\psi}]\\ &= \{x,G\}^{\psi}-\{y,F\}^{\psi}-\mathcal{V}[\psi]\\ &= \psi\frac{\partial G}{\partial y}+\psi\frac{\partial F}{\partial x}-\Big(F\frac{\partial\psi}{\partial x}+G\frac{\partial\psi}{\partial y}\Big),\end{aligned}$$

so that, $\delta^1(\mathcal{V})$, viewed as an element of $\mathcal{A}$ is, using the notations of Paragraph 4.1.1,

$$\psi\Big(\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}\Big)-\Big(F\frac{\partial\psi}{\partial x}+G\frac{\partial\psi}{\partial y}\Big)=\psi\,\mathrm{Div}(\vec{F}\,)-\vec{F}\cdot\vec{\nabla}\psi,$$

so that the Poisson cohomology complex of $(\mathcal{A},\{\cdot\,,\cdot\}^{\psi})$ can be written as follows:

$$\begin{array}{ccccccc} 0\longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{A}^2 & \longrightarrow & \mathcal{A} & \longrightarrow 0\\ & H & \longmapsto & \psi\vec{\mathcal{H}}_H & & & \\ & & & \vec{F} & \longmapsto & \psi\,\mathrm{Div}(\vec{F}\,)-\vec{F}\cdot\vec{\nabla}\psi & \end{array}\tag{4.15}$$

We now express the Poisson cohomology spaces of $(\mathcal{A},\{\cdot\,,\cdot\}^{\psi})$, denoted by $H^k(\mathcal{A},\psi)$, in terms of elements of $\mathcal{A}$ and $\mathcal{A}^2$:

$$H^0(\mathcal{A},\psi) \simeq \left\{H \in \mathcal{A} \mid \psi \vec{\mathcal{H}}_H = \vec{0}\right\},$$

$$H^1(\mathcal{A},\psi) \simeq \frac{\left\{\vec{F} \in \mathcal{A}^2 \mid \psi \operatorname{Div}(\vec{F}\,) - \vec{F}\cdot\vec{\nabla}\psi = 0\right\}}{\left\{\psi \vec{\mathcal{H}}_H \mid H \in \mathcal{A}\right\}},$$

$$H^2(\mathcal{A},\psi) \simeq \frac{\mathcal{A}}{\left\{\psi \operatorname{Div}(\vec{F}\,) - \vec{F}\cdot\vec{\nabla}\psi \mid \vec{F} \in \mathcal{A}^2\right\}}.$$

It is clear that we have $H^0(\mathcal{A},\psi) = \mathcal{A}$, if $\psi = 0$ and $H^0(\mathcal{A},\psi) = \mathbf{F}$, if $\psi \neq 0$.

Remark 4.7. Suppose that $\psi \in \mathbf{F}$ is a non zero constant polynomial. Then, the Poisson cohomology spaces of $(\mathcal{A}, \{\cdot\,,\cdot\}^\psi)$ become

$$H^0(\mathcal{A},\psi) \simeq \left\{H \in \mathcal{A} \mid \vec{\mathcal{H}}_H = \vec{0}\right\}, \quad H^1(\mathcal{A},\psi) \simeq \frac{\left\{\vec{F} \in \mathcal{A}^2 \mid \operatorname{Div}(\vec{F}\,) = 0\right\}}{\left\{\vec{\mathcal{H}}_H \mid H \in \mathcal{A}\right\}},$$

$$H^2(\mathcal{A},\psi) \simeq \frac{\mathcal{A}}{\left\{\operatorname{Div}(\vec{F}\,) \mid \vec{F} \in \mathcal{A}^2\right\}},$$

and this is an illustration of the well-known result that says, in the symplectic case, that the Poisson cohomology is isomorphic to the de Rham cohomology. By observing the de Rham complex in dimension two (4.11), we see indeed that, in the case where ψ is a non zero constant polynomial (and the corresponding Poisson bracket is the classical symplectic structure $\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$), the de Rham complex is exactly equivalent to the Poisson cohomology complex. In this context, according to Proposition 4.1, the de Rham cohomology, and so the Poisson cohomology, is trivial:

$$H^0(\mathcal{A},\psi) \simeq \mathbf{F}, \quad H^1(\mathcal{A},\psi) \simeq \{0\}, \quad H^2(\mathcal{A},\psi) \simeq \{0\}; \quad \text{ if } \psi \in \mathbf{F} - \{0\}.$$

We will now assume that the polynomial ψ is a non constant weight homogeneous polynomial. We denote by ϖ_1 and ϖ_2 the corresponding weights of the two variables x and y. We recall that $\varpi(\psi)$ denotes the (weighted) degree of ψ.

Remark 4.8. As in dimension three, for the Poisson variety $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$, we point out that, if ψ is a weight homogeneous polynomial of $\mathcal{A} = \mathbf{F}[x,y]$, the Poisson coboundary operator, associated to the Poisson algebra $(\mathcal{A}, \{\cdot\,,\cdot\}^\psi)$, is a weight homogeneous operator and satisfies:

$$P \in \mathfrak{X}^k(\mathcal{A})_i \Rightarrow \delta^k(P) \in \mathfrak{X}^{k+1}(\mathcal{A})_{i+N'(\psi)},$$

where $N'(\psi) = \varpi(\psi) - \varpi_1 - \varpi_2 \in \mathbf{Z}$ is exactly the weight of the Poisson biderivation $\{\cdot\,,\cdot\}^\psi \in \mathfrak{X}^2(\mathcal{A})$. This fact allows us to work "degree by degree", in order to compute the Poisson cohomology spaces of $(\mathbf{F}^2, \{\cdot\,,\cdot\}^\psi)$, as we have done for $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$.

In their paper [54], C. Roger and P. Vanhaecke show that $H^1(\mathcal{A},\psi)$ and $H^2(\mathcal{A},\psi)$ are infinite dimensional if and only if ψ is not square-free. In our discussion, we will assume that ψ is square free, so that the Poisson cohomology spaces are of finite dimension.

First, we will determine the first Poisson cohomology space $H^1(\mathcal{A},\psi)$. To do this, we need the following result.

Lemma 4.9. *Let $\psi \in \mathcal{A} = \mathbf{F}[x,y]$ be a square-free weight homogeneous polynomial. Suppose that $K \in \mathcal{A}$ is a polynomial satisfying:*

$$\vec{\mathcal{H}}_K \cdot \vec{\nabla}\psi = 0.$$

Then $K \in \bigoplus_{r\in\mathbf{N}} \mathbf{F}\psi^r$.

Proof. Let $K \in \mathbf{F}[x,y] - \{0\}$, such that $\vec{\mathcal{H}}_K \cdot \vec{\nabla}\psi = 0$. We suppose that K is weight homogeneous. The proof of this result will be analogous to the one of Proposition 3.11. We write $K = H\psi^r$, with $r \in \mathbf{N}$ and $H \in \mathcal{A} - \{0\}$, a weight homogeneous polynomial which is not divisible by ψ. Then, we have:

$$\vec{\mathcal{H}}_K = \psi^r \vec{\mathcal{H}}_H + rH\psi^{r-1}\vec{\mathcal{H}}_\psi,$$

so that, $\vec{\mathcal{H}}_H \cdot \vec{\nabla}\psi = 0 = -\vec{\nabla}H \cdot \vec{\mathcal{H}}_\psi$, according to Formula (4.3). Using the exactness of the Koszul complex associated to ψ, in Proposition 4.6, there exists a polynomial $J \in \mathcal{A} = \mathbf{F}[x,y]$ such that $\vec{\nabla}H = J\vec{\nabla}\psi$. Euler's Formula (4.4) then yields

$$\varpi(H)H = \varpi(\psi)J\psi,$$

so that, $\varpi(H) = 0$, because ψ does not divide H. We have obtained that $K = H\psi^r \in \mathbf{F}\psi^r$, hence the result. □

Proposition 4.10. *If $\psi \in \mathcal{A} = \mathbf{F}[x,y]$ is a square-free weight homogeneous polynomial, then the first Poisson cohomology space of the Poisson variety $(\mathbf{F}^2, \{\cdot\,,\cdot\}^\psi)$ is given by:*

$$H^1(\mathcal{A},\psi) \simeq \mathcal{A}_{N'(\psi)}\,\vec{e}_\varpi \oplus \mathbf{F}\vec{\mathcal{H}}_\psi,$$

where $\mathcal{A}_{N'(\psi)}$ is the $\mathbf{F}$-vector space of all weight homogeneous polynomials of $\mathcal{A} = \mathbf{F}[x,y]$, of degree equal to $N'(\psi) = \varpi(\psi) - w_1 - \varpi_2$.

Proof. Let $\vec{F} \in \mathcal{A}^2$ be an element of $Z^1(\mathcal{A},\psi)$, i.e., satisfying:

$$\delta^1(\vec{F}\,) = \psi\,\mathrm{Div}(\vec{F}\,) - \vec{F}\cdot\vec{\nabla}\psi = 0. \tag{4.16}$$

Considering Remark 4.8, assume that $\vec{F} \in \mathfrak{X}^1(\mathcal{A})$ is weight homogeneous of degree $r \in \mathbf{Z}$ (i.e., according to the decompositions (4.6), $\vec{F} \in \mathcal{A}_{r+\varpi_1} \times \mathcal{A}_{r+\varpi_2}$). Using Euler's Formula (3.5), the cocycle equation (4.16) can be written as:

$$\Big(\frac{1}{\varpi(\psi)}\,\mathrm{Div}(\vec{F})\,\vec{e}_\varpi - \vec{F}\Big)\cdot\vec{\nabla}\psi = 0,$$

so that, according to the exactness of the Koszul complex in Proposition 4.6, there exists $G \in \mathcal{A}$, such that:

$$\frac{1}{\varpi(\psi)} \operatorname{Div}(\vec{F}\,)\, \vec{e}_\varpi - \vec{F} = G\, \vec{\mathcal{H}}_\psi. \tag{4.17}$$

• If $r = \varpi(\vec{F}) = \varpi(\psi) - \varpi_1 - \varpi_2 = N'(\psi)$, then

$$\operatorname{Div}(\vec{F}\,) \in \mathcal{A}_{N'(\psi)}.$$

According to the decompositions (4.6), we have $\vec{\mathcal{H}}_\psi \in \mathfrak{X}^1(\mathcal{A})_{\varpi(\psi)-\varpi_1-\varpi_2}$, and

$$\varpi(G) = \varpi(\vec{F}\,) - \varpi(\vec{\mathcal{H}}_\psi) = \varpi(\vec{F}\,) - \varpi(\psi) + \varpi_1 + \varpi_2 = 0,$$

so that

$$\vec{F} = \frac{1}{\varpi(\psi)} \operatorname{Div}(\vec{F}\,)\, \vec{e}_\varpi - G\, \vec{\mathcal{H}}_\psi \in \mathcal{A}_{N'(\psi)}\, \vec{e}_\varpi + \mathbf{F}\, \vec{\mathcal{H}}_\psi.$$

• Now, suppose that $\varpi(\vec{F}) \neq N'(\psi)$.

We compute the divergence of the equation (4.17) and, by using Euler's Formula (4.5), Formulas (4.1), (4.2) and (4.3), we obtain:

$$\Big(\frac{r + \varpi_1 + \varpi_2}{\varpi(\psi)} - 1\Big) \operatorname{Div}(\vec{F}\,) = \vec{\nabla} G \cdot \vec{\mathcal{H}}_\psi = -\vec{\nabla}\psi \cdot \vec{\mathcal{H}}_G = -\operatorname{Div}(\psi \vec{\mathcal{H}}_G), \tag{4.18}$$

so that, according to the exactness of the Koszul complex (Proposition 4.6), there exists a weight homogeneous $K \in \mathcal{A}$ such that

$$\Big(\frac{r + \varpi_1 + \varpi_2}{\varpi(\psi)} - 1\Big) \vec{F} + \psi \vec{\mathcal{H}}_G = \vec{\mathcal{H}}_K. \tag{4.19}$$

If $K \in \mathbf{F}$, we have $\vec{F} \in \mathbf{F}\psi\vec{\mathcal{H}}_G \subset B^1(\mathcal{A}, \psi)$ $(r \neq \varpi(\psi) - \varpi_1 - \varpi_2)$. Let us now suppose that K is not constant. By computing the inner product of (4.19) with $\vec{\nabla}\psi$, by using the cocycle equation (4.16) and the string (4.18), we get:

$$\begin{aligned}
\vec{\mathcal{H}}_K \cdot \vec{\nabla}\psi &= \Big(\frac{r + \varpi_1 + \varpi_2}{\varpi(\psi)} - 1\Big) \vec{F} \cdot \vec{\nabla}\psi + \psi \vec{\mathcal{H}}_G \cdot \vec{\nabla}\psi \\
&= \Big(\frac{r + \varpi_1 + \varpi_2}{\varpi(\psi)} - 1\Big) \psi \operatorname{Div}(\vec{F}\,) + \psi \vec{\mathcal{H}}_G \cdot \vec{\nabla}\psi \\
&= -\psi \vec{\nabla}\psi \cdot \vec{\mathcal{H}}_G + \psi \vec{\mathcal{H}}_G \cdot \vec{\nabla}\psi \\
&= 0
\end{aligned}$$

According to Lemma 4.9, there exist $c \in \mathbf{F}$ and $s \in \mathbf{N}^*$ such that $K = c\psi^s$, so that $\vec{\mathcal{H}}_K = cs\psi^{s-1}\vec{\mathcal{H}}_\psi$. Equation (4.19) leads to $\varpi(K) = \varpi(\vec{F}) + \varpi_1 + \varpi_2 \neq \varpi(\psi)$ (because $\varpi(\vec{F}) \neq N'(\psi)$). That implies $s \geq 2$ and (4.19) yields:

$$\vec{F} = \psi\, \vec{\mathcal{H}}_F \in B^1(\mathcal{A}, \psi),$$

with $F = \dfrac{\varpi(\psi)}{r - N'(\psi)}(-G + \dfrac{cs}{s-1}\psi^{s-1})$. We have then obtained, by considering both cases $r = N'(\psi)$ and $r \neq N'(\psi)$, that

$$Z^1(\mathcal{A}, \psi) \subset B^1(\mathcal{A}, \psi) + \mathcal{A}_{N'(\psi)}\, \vec{e}_{\varpi} + \mathbf{F}\vec{\mathcal{H}}_{\psi}.$$

The other inclusion is obvious. If $H \in \mathcal{A}_{N'(\psi)}$, we use indeed (4.5) and (4.4) to obtain:

$$\delta^1(H\,\vec{e}_{\varpi}) = \psi\,\mathrm{Div}(H\,\vec{e}_{\varpi}) - H\,\vec{e}_{\varpi}\cdot\vec{\nabla}\psi = 0,$$

and we use Formula (4.2) to get:

$$\delta^1(\vec{\mathcal{H}}_{\psi}) = \psi\,\mathrm{Div}(\vec{\mathcal{H}}_{\psi}) - \vec{\mathcal{H}}_{\psi}\cdot\vec{\nabla}\psi = 0.$$

Now, it remains to show that the decomposition of $Z^1(\mathcal{A}, \psi)$ obtained is a direct one. To do this, let us consider a polynomial $H \in \mathcal{A}_{N'(\psi)}$, an element $F \in \mathcal{A}$ and $c \in \mathbf{F}$ satisfying:

$$H\,\vec{e}_{\varpi} + c\vec{\mathcal{H}}_{\psi} = \delta^0(F) = \psi\,\vec{\mathcal{H}}_F.$$

Here, we have $\varpi(H) = N'(\psi) = \varpi(\psi) + \varpi(F) - (\varpi_1 + \varpi_2)$, so that $\varpi(F) = 0$ and $F \in \mathbf{F}$. Then, we have $\vec{\mathcal{H}}_F = \vec{0}$ and

$$H\,\vec{e}_{\varpi} + c\vec{\mathcal{H}}_{\psi} = \vec{0}.$$

By computing the divergence, we get:

$$(N'(\psi) + \varpi_1 + \varpi_2)\,H = \varpi(\psi)\,H = 0,$$

thus $H = 0$, $c = 0$ and the sum is direct, we have exactly:

$$Z^1(\mathcal{A}, \psi) = B^1(\mathcal{A}, \psi) \oplus \mathcal{A}_{N'(\psi)}\,\vec{e}_{\varpi} \oplus \mathbf{F}\vec{\mathcal{H}}_{\psi},$$

that gives the desired result. □

Now, let us turn out with the second Poisson cohomology space of the Poisson variety $(\mathbf{F}^2, \{\cdot\,,\cdot\}^{\psi})$.

Proposition 4.11. *If $\psi \in \mathcal{A} = \mathbf{F}[x, y]$ is a square-free weight homogeneous polynomial, then the second Poisson cohomology space of the Poisson variety $(\mathbf{F}^2, \{\cdot\,,\cdot\}^{\psi})$ is given by:*

$$H^2(\mathcal{A}, \psi) \simeq \mathcal{A}_{N'(\psi)}\,\psi \;\oplus\; \frac{\mathbf{F}[x, y]}{\left\langle \dfrac{\partial\psi}{\partial x}, \dfrac{\partial\psi}{\partial y} \right\rangle},$$

where $\mathcal{A}_{N'(\psi)}$ is the $\mathbf{F}$-vector space of all the weight homogeneous polynomials of $\mathcal{A}$, of degree equal to $N'(\psi) = \varpi(\psi) - \varpi_1 - \varpi_2$.

Proof. We recall from Lemma 4.3, that, as ψ is supposed to be a square-free weight homogeneous polynomial of $\mathbf{F}[x,y]$, the $\mathbf{F}$-vector space

$$\mathcal{A}_{sing}(\psi) := \frac{\mathbf{F}[x,y]}{\left\langle \dfrac{\partial\psi}{\partial x}, \dfrac{\partial\psi}{\partial y} \right\rangle}$$

is of finite dimension. Let us denote by μ_ψ the dimension of this $\mathbf{F}$-vector space (i.e., the Milnor number of ψ) and by $v_0 = 1, v_1, \ldots, v_{\mu_\psi - 1}$ a family of weight homogeneous polynomials of $\mathbf{F}[x,y]$, whose images in $\mathcal{A}_{sing}(\psi)$ give a $\mathbf{F}$-basis of this $\mathbf{F}$-vector space.

Let $F \in \mathcal{A} = \mathbf{F}[x,y]$ be a weight homogeneous polynomial of degree $r \in \mathbf{N}$. According to the definition of $v_0, \ldots, v_{\mu_\psi - 1}$, we have the existence of a weight homogeneous element $\vec{G} \in \mathcal{A}^2$ and of constants $\lambda_j \in \mathbf{F}$, for $0 \le j \le \mu_\psi - 1$ satifying

$$F = \vec{G} \cdot \vec{\nabla}\psi + \sum_{j=0}^{\mu_\psi - 1} \lambda_j v_j.$$

We have $\vec{G} \in \mathfrak{X}^1(\mathcal{A})_{r-\varpi(\psi)}$ and $\varpi\left(\mathrm{Div}(\vec{G})\right) = r - \varpi(\psi)$.

• Assume that $r \neq 2\varpi(\psi) - \varpi_1 - \varpi_2 = N'(\psi) + \varpi(\psi)$.

For any weight homogeneous polynomial $H \in \mathcal{A}$, according to Euler's Formulas (4.5) and (4.4), we have:

$$\begin{aligned}
\delta^1(H\,\vec{e}_\varpi) &= \psi\,\mathrm{Div}(H\,\vec{e}_\varpi) - H\,\vec{e}_\varpi \cdot \vec{\nabla}\psi \\
&= (\varpi(H) + \varpi_1 + \varpi_2 - \varpi(\psi))\,\psi\,H \\
&= (\varpi(H) - N'(\psi))\,\psi\,H.
\end{aligned}$$

So that, we have

$$\delta^1\Big(\frac{1}{r - \varpi(\psi) - N'(\psi)}\,\mathrm{Div}(\vec{G})\vec{e}_\varpi - \vec{G}\Big) = \psi\,\mathrm{Div}(\vec{G}) - \delta^1(\vec{G}) = \vec{G}\cdot\vec{\nabla}\psi \in B^2(\mathcal{A},\psi),$$

and $F \in B^2(\mathcal{A},\psi) + \sum_{j=0}^{\mu_\psi-1} \mathbf{F} v_j$, in this case.

• Now, let us suppose that $r = N'(\psi) + \varpi(\psi)$.

Then, $\varpi(\mathrm{Div}(\vec{G}\,)) = r - \varpi(\psi) = N'(\psi)$ and

$$\vec{G}\cdot\vec{\nabla}\psi = -\delta^1(\vec{G}\,) + \psi\,\mathrm{Div}(\vec{G}\,) \in B^2(\mathcal{A},\psi) + \mathcal{A}_{N'(\psi)}\,\psi$$

and, according to the above writing of F, we have

$$F \in B^2(\mathcal{A},\psi) + \mathcal{A}_{N'(\psi)}\,\psi + \sum_{j=0}^{\mu_\psi-1} \mathbf{F}\,v_j.$$

By studying both cases $r \neq N'(\psi) + \varpi(\psi)$ and $r = N'(\psi) + \varpi(\psi)$, we have obtained the equality

$$\mathcal{A} = B^2(\mathcal{A}, \psi) + \mathcal{A}_{N'(\psi)}\,\psi + \sum_{j=0}^{\mu_\psi - 1} \mathbf{F} v_j. \tag{4.20}$$

It remains to show that this sum is direct. For this purpose, let us consider $H \in \mathcal{A}_{N'(\psi)}$, $\vec{F} \in \mathcal{A}^2$ and some elements $\lambda_j \in \mathcal{A}$, for $0 \leq j \leq \mu_\psi - 1$, such that

$$\delta^1(\vec{F}\,) = \psi\,\mathrm{Div}(\vec{F}\,) - \vec{F} \cdot \vec{\nabla}\psi = H\,\psi + \sum_{j=0}^{\mu_\psi - 1} \lambda_j v_j. \tag{4.21}$$

As H is a weight homogeneous polynomial of degree $N'(\psi)$, we assume that $\vec{F} \in \mathfrak{X}^1(\mathcal{A})_{N'(\psi)}$.

With the help of Euler's Formula (4.4), Equality (4.21) leads to

$$\sum_{j=0}^{\mu_\psi - 1} \lambda_j v_j \in \left\langle \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y} \right\rangle,$$

which yields, according to the definition of the v_j, $\lambda_j = 0$, for all $0 \leq j \leq \mu_{\psi-1}$. Now, with again the help of Euler's Formula (4.4) and of the exactness of the Koszul complex in Proposition 4.6, we have the further equalities:

$$\psi\,\mathrm{Div}(\vec{F}\,) - H\,\psi = \vec{F} \cdot \vec{\nabla}\psi,$$
$$\frac{1}{\varpi(\psi)}\Big(\mathrm{Div}(\vec{F}\,) - H\Big)\vec{e}_\varpi = \vec{F} + G\vec{\mathcal{H}}_\psi,$$

where $G \in \mathcal{A}$ is a weight homogeneous polynomial. In fact, $\varpi(G) = \varpi(H) - \varpi(\psi) + \varpi_1 + \varpi_2 = 0$, by hypothesis, so that, $G \in \mathbf{F}$. We compute the divergence of the latter equality (with Formulas (4.5) and (4.2)), and we obtain:

$$\frac{N'(\psi) + \varpi_1 + \varpi_2}{\varpi(\psi)}\Big(\mathrm{Div}(\vec{F}\,) - H\Big) = \mathrm{Div}(\vec{F}\,) - H = \mathrm{Div}(\vec{F}\,),$$

that gives $H = 0$. We then have obtained that the sum (4.20) is a direct one, and so we have proved the promided result. □

4.2.2 Poisson homology of $(\mathbf{F}^2, \{\cdot\,,\cdot\}^\psi)$

We consider the affine space of dimension two, $\mathbf{F}^2$, its algebra of regular functions $\mathcal{A} = \mathbf{F}[x, y]$ and a weight homogeneous Poisson structure on $\mathcal{A}$

$$\{\cdot\,,\cdot\}^\psi = \psi\,\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y},$$

where $\psi \in \mathcal{A}$ is square-free. Under the identifications of the spaces of all Kähler differentials in paragraph 4.1.2, we are able to write the Poisson homology complex in terms of elements of $\mathcal{A}$ and $\mathcal{A}^2$:

$$\{0\} \longrightarrow \Omega^2(\mathcal{A}) \simeq \mathcal{A} \xrightarrow{\partial_2} \Omega^1(\mathcal{A}) \simeq \mathcal{A}^2 \xrightarrow{\partial_1} \Omega^0(\mathcal{A}) \simeq \mathcal{A} \xrightarrow{\partial_0} \{0\}$$

To do this, we have to consider the definition of ∂_k in Paragraph 2.2.2 and the identifications given in the paragraph 4.1.2. For $F \in \mathcal{A} \simeq \Omega^0(\mathcal{A})$, $\partial_0(F) = 0$ and, for $\vec{F} = (F, G) \in \mathcal{A}^2 \simeq \Omega^1(\mathcal{A})$, we have $\partial_1(\vec{F}\ ) \in \Omega^0(\mathcal{A}) \simeq \mathcal{A}$, given by:

$$\partial_1(\vec{F}\ ) = \{F, x\}^\psi + \{G, y\}^\psi = \psi\left(-\frac{\partial F}{\partial y} + \frac{\partial G}{\partial x}\right).$$

Then, for $F \in \Omega^2(\mathcal{A}) \simeq \mathcal{A}$, $\partial_2(F) \in \Omega^1(\mathcal{A})$ is given by:

$$\begin{aligned}\partial_2(F) &= \{F, x\}^\psi \,\mathsf{d}y - \{F, y\}^\psi \,\mathsf{d}x - F\,\mathsf{d}\,\{x, y\}^\psi \\ &= -\psi\frac{\partial F}{\partial y}\mathsf{d}y - \psi\frac{\partial F}{\partial x}\mathsf{d}x - F\mathsf{d}\psi \\ &= -\mathsf{d}(\psi F)\end{aligned}$$

but under the identifications mentioned above, we see $\partial_2(F) \in \Omega^1(\mathcal{A}) \simeq \mathcal{A}^2$ as the element $-\vec{\nabla}(F\psi) := -\left(\frac{\partial(F\psi)}{\partial x}, \frac{\partial(F\psi)}{\partial y}\right) \in \mathcal{A}^2$. Finally, we obtain

$$\begin{aligned}&\partial_0(F) = 0, \quad \text{for } F \in \mathcal{A} \simeq \Omega^0(\mathcal{A}),\\ &\partial_1(\vec{F}\ ) = \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right)\psi, \quad \text{for } \vec{F} = (F, G) \in \mathcal{A}^2 \simeq \Omega^1(\mathcal{A}),\\ &\partial_2(F) = -\vec{\nabla}(F\psi), \quad \text{for } F \in \mathcal{A} \simeq \Omega^2(\mathcal{A}).\end{aligned} \tag{4.22}$$

These writings lead to the writing of the (classical) Poisson homology spaces in dimension two:

$$\begin{aligned}H_0(\mathcal{A}, \psi) &\simeq \frac{\mathcal{A}}{\left\{\left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right)\psi \mid F, G \in \mathcal{A}\right\}},\\ H_1(\mathcal{A}, \psi) &\simeq \frac{\left\{(F, G) \in \mathcal{A}^2 \mid \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right)\psi = 0\right\}}{\left\{\vec{\nabla}(K\psi) \mid K \in \mathcal{A}\right\}},\\ H_2(\mathcal{A}, \psi) &\simeq \left\{F \in \mathcal{A} \mid \vec{\nabla}(F\psi) = 0\right\}.\end{aligned} \tag{4.23}$$

Now we can explicitly determine these spaces, in terms of elements of $\mathcal{A}$ or $\mathcal{A}^2$, by supposing that ψ is a non constant, weight homogeneous, square free polynomial (See the thesis of P. Monnier, where these computations are done).

Remark 4.12. We point out that, in the symplectic case (See Remark 4.7) where $\psi \in \mathbf{F}$ is a non zero constant polynomial, the Poisson homology spaces are given by:

$$H_0(\mathcal{A},\psi) \simeq \frac{\mathcal{A}}{\left\{\left(\frac{\partial G}{\partial x}-\frac{\partial F}{\partial y}\right) \mid F,G\in\mathcal{A}\right\}} \simeq \{0\},$$
$$H_1(\mathcal{A},\psi) \simeq \frac{\left\{(F,G)\in\mathcal{A}^2 \mid \left(\frac{\partial G}{\partial x}-\frac{\partial F}{\partial y}\right)=0\right\}}{\left\{\vec{\nabla}K \mid K\in\mathcal{A}\right\}} \simeq \{0\},$$
$$H_2(\mathcal{A},\psi) \simeq \left\{F\in\mathcal{A} \mid \vec{\nabla}F=0\right\} \simeq \mathbf{F}.$$

(where for $H_1(\mathcal{A},\psi)$, we have use the exactness of the de Rahm complex, in Proposition 4.1.)

1. As we suppose that ψ is non constant, we see that
$$\boxed{H_2(\mathcal{A},\psi)\simeq\{0\}}.$$
2. Because ψ is not equal to zero and $\mathcal{A} = \left\{\left(\frac{\partial F}{\partial y}-\frac{\partial G}{\partial x}\right) \mid F,G\in\mathcal{A}\right\}$, we have:
$$\boxed{H_0(\mathcal{A},\psi)\simeq\mathcal{A}/\langle\psi\rangle}.$$
3. Finally, let us consider the space $H_1(\mathcal{A},\psi)$. First, for $F,G\in\mathcal{A}$, according to the Proposition 4.1 that gives the exactness of the de Rham complex, we have that:
$$\left(\frac{\partial F}{\partial y}-\frac{\partial G}{\partial x}\right)\psi=0 \iff \frac{\partial F}{\partial y}-\frac{\partial G}{\partial x}=0 \iff (F,G)=\vec{\nabla}H,$$
with $H\in\mathcal{A}$. So that, we have
$$H_1(\mathcal{A},\psi)\simeq\frac{\left\{\vec{\nabla}H \mid H\in\mathcal{A}\right\}}{\left\{\vec{\nabla}(F\psi) \mid F\in\mathcal{A}\right\}}.$$
Moreover, let us suppose that $H,F\in\mathcal{A}$ are two weight homogeneous polynomials. Euler's Formula (4.4) allows one to write:
$$\vec{\nabla}H=\vec{\nabla}(F\psi) \Longrightarrow \vec{\nabla}H\cdot\vec{e}_\varpi=\vec{\nabla}(F\psi)\cdot\vec{e}_\varpi$$
$$\iff \varpi(H)\,H=(\varpi(F)+\varpi(\psi))\,F\psi,$$
so that $H\in\mathbf{F}$ (and $F=0$) or $H\in\langle\psi\rangle$. We conclude that
$$\boxed{H_1(\mathcal{A},\psi)\simeq\frac{\langle x,y\rangle}{\langle\psi\rangle}}.$$

Remark 4.13 (Modular derivation). The results of Poisson cohomology and homology obtained for the Poisson variety $(\mathbf{F}^2, \{\cdot\,,\cdot\}^\psi)$ will permit us to illustrate a different configuration of the unimodular property than in the case of the Poisson variety $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$ in Chapter 3 (see Remark (3.24)).

Indeed, the Poisson variety $(\mathbf{F}^2, \{\cdot\,,\cdot\}^\psi)$ is unimodular if and only if $\psi \in \mathbf{F}$, as the modular derivation $\mathrm{Div}(\{\cdot\,,\cdot\}^\psi)$ is, according to (4.8), given by

$$*(\mathrm{Div}(\{\cdot\,,\cdot\}^\psi)) = \mathsf{d} * (\{\cdot\,,\cdot\}^\psi) = \mathsf{d}\psi.$$

We have seen in Remark 4.7 that the case of $\psi \in \mathbf{F}^*$ is the symplectic one, where the Poisson cohomology is isomorphic to the de Rham one and we have seen in Remark 4.12 that in this case, we have a duality $(H^k(\mathcal{A}) \simeq H_{2-k}(\mathcal{A}))$ between the Poisson cohomology spaces and the Poisson homology spaces.

If we exclude this case, we have no chance to have a modular derivation equal to zero and to apply Proposition 2.28. We point out that, according to the results we have obtained, if ψ is not a constant polynomial, then the Poisson cohomology space $H^k(\mathcal{A})$ is not isomorphic to the Poisson homology space $H_{2-k}(\mathcal{A})$ (for $k = 0, 1, 2$).

4.3 Cohomology of the singular surface $\mathcal{F}_\varphi$

In this section, we consider an other Poisson variety of dimension two than $(\mathbf{F}^2, \{\cdot\,,\cdot\}^\psi)$ as we want to study a singular affine surface in $\mathbf{F}^3$. To do this, we consider as in Section 3.2, a weight homogeneous polynomial $\varphi \in \mathcal{A} := \mathbf{F}[x, y, z]$, with an isolated singularity and the singular surface defined by its zero locus $\mathcal{F}_\varphi : \{\varphi = 0\} \subset \mathbf{F}^3$. Our purpose is to equip this variety with a Poisson structure as regular as possible and to compute the corresponding Poisson (co)homology.

4.3.1 The Poisson complex of the singular surface $\mathcal{F}_\varphi$

In this paragraph, we want to equip the algebra of regular functions on the surface $\mathcal{F}_\varphi$, namely, the algebra $\mathcal{A}_\varphi := \mathbf{F}[x, y, z]/\langle\varphi\rangle$, with a Poisson bracket and to study the Poisson cochains, i.e., the skew-symmetric multi-derivations of $\mathcal{F}_\varphi$. To do this, we will first establish a relation between the skew-symmetric multi-derivations of $\mathbf{F}[x_1, \ldots, x_n]$ and those of the algebra of a general affine variety, $\mathbf{F}[x_1, \ldots, x_n]/\mathcal{I}$.

Multi-derivations of $\mathbf{F}[x_1, \ldots, x_n]$ and $\mathbf{F}[x_1, \ldots, x_n]/\mathcal{I}$

In this paragraph, we will give a result that will help us to write the skew-symmetric multi-derivations of an algebra of regular functions on an affine variety $M \subset \mathbf{F}^n$. In order to simplify the notations, we will, in this paragraph, denote with a bar the objects relative to a quotient, as explained now.

Let M be an affine variety and let $\bar{\mathcal{A}}$ be the algebra of regular functions on M. One can write $\bar{\mathcal{A}}$ as the quotient $\bar{\mathcal{A}} := \mathbf{F}[x_1, \ldots, x_n]/\mathcal{I}$ of the polynomial algebra $\mathcal{A} := \mathbf{F}[x_1, \dots, x_n]$ and an ideal $\mathcal{I}$ of $\mathcal{A}$ (see the paragraph 2.1.1). For $F \in \mathcal{A} = \mathbf{F}[x_1, \dots, x_n]$, we denote by $\bar{F}$ the image of F in the quotient algebra $\bar{\mathcal{A}}$, so that, $\bar{\mathcal{A}}$ is generated by the elements $\bar{x}_1, \dots, \bar{x}_n$. We also denote by $\wp$ the projection map $\wp : \mathcal{A} \mapsto \bar{\mathcal{A}} : F \to \wp(F) = \bar{F}$.

We want to study the skew-symmetric multi-derivations of $\bar{\mathcal{A}}$. We will see that every skew-symmetric multi-derivation on $\bar{\mathcal{A}}$ is induced by a skew-symmetric multi-derivation of $\mathcal{A}$.

First, we recall that, according to Proposition 2.15, a skew-symmetric k-derivation of $\mathcal{A} = \mathbf{F}[x_1, \dots, x_n]$, $P \in \mathfrak{X}^k(\mathcal{A})$, is completely defined by its values on the generators of the algebra $\mathcal{A}$. More precisely, for $F_1, \dots, F_k \in \mathcal{A}$, we have seen that:

$$P[F_1, \dots, F_k] = \sum_{1 \leq i_1, \dots, i_k \leq n} P[x_{i_1}, \dots, x_{i_k}] \frac{\partial F_1}{\partial x_{i_1}} \cdots \frac{\partial F_k}{\partial x_{i_k}}. \tag{4.24}$$

Now, we would like to generalize this formula to the skew-symmetric k-derivations of $\bar{\mathcal{A}}$, that is to say, we want to obtain an analogous formula as (4.24) for the skew-symmetric k-derivations of $\bar{\mathcal{A}}$. The problem is that the sum in the right side of the equality (4.24) does not clearly give a well defined map on $\bar{\mathcal{A}}^k$.

Proposition 4.14. *Let M be an affine variety and let $\bar{A}$ be its algebra of regular functions. Let $\bar{P} \in \mathfrak{X}^k(\bar{\mathcal{A}})$ be a skew-symmetric k-derivation of $\bar{\mathcal{A}}$. For all $F_1, \dots, F_k \in \mathcal{A}$, we have:*

$$\bar{P}[\bar{F}_1, \dots, \bar{F}_k] = \sum_{1 \leq i_1, \dots, i_k \leq n} \bar{P}[\bar{x}_{i_1}, \dots, \bar{x}_{i_k}] \overline{\frac{\partial F_1}{\partial x_{i_1}}} \cdots \overline{\frac{\partial F_k}{\partial x_{i_k}}}. \tag{4.25}$$

Proof. The proof will be inspired from the proof of Proposition 2.3. Let $\bar{P} \in \mathfrak{X}^k(\bar{\mathcal{A}})$. As $\bar{P}$ and the expression given on the right hand side of (4.25) are k-linear in $F_1, \dots, F_k$, it suffices to show this equality for $F_1, \dots, F_k$, monomials in $x_1, \dots, x_n$. We will show Formula (4.25) by recursion.

First, if there exists $1 \leq l \leq k$, such that $\deg(F_l) = 0$, i.e., $F_l = 1$, then, $\bar{P}[\bar{F}_1, \dots, \bar{F}_k] = 0$. Moreover,if $F_1, \dots, F_k$ are all of degree 1, namely, $F_r = x_{i_r}$, $1 \leq r \leq k$, then, it is clear that the equality (4.25) holds.

Now, assume that Formula (4.25) holds for all $F_1, \dots, F_k \in \mathcal{A}$, such that $\deg(F_1) + \cdots + \deg(F_k) \leq m$, for some $m \geq k$. We show now that it holds for all $F_1, \dots, F_k$ such that $\deg(F_1) + \cdots + \deg(F_k) = m + 1$. Let $F_1, \dots, F_k$ be non-constant monomials, such that $\deg(F_1) + \cdots + \deg(F_k) = m + 1$. Suppose that $\deg(F_1) > 1$. Then, there exists $G, H \in \mathcal{A}$, two non zero polynomials, with $\deg(G) < \deg(F_1)$ and $\deg(H) < \deg(F_1)$, such that $F_1 = GH$. Since $\bar{P}$ is a k-derivation and in view of the recursion hypothesis, we have that

$$\begin{aligned}\bar{P}[\bar{F}_1, \bar{F}_2, \dots, \bar{F}_k] &= \bar{P}[\overline{GH}, \bar{F}_2, \dots, \bar{F}_k] = \bar{P}[\bar{G}\bar{H}, \bar{F}_2, \dots, \bar{F}_k] \\ &= \bar{G}\,\bar{P}[\bar{H}, \bar{F}_2, \dots, \bar{F}_k] + \bar{H}\,\bar{P}[\bar{G}, \bar{F}_2, \dots, \bar{F}_k]\end{aligned}$$

$$
\begin{aligned}
&= \bar G \sum_{1\le i_1,\dots,i_k\le n} \bar P[\bar x_{i_1},\dots,\bar x_{i_k}]\,\overline{\frac{\partial H}{\partial x_{i_1}}}\,\overline{\frac{\partial F_2}{\partial x_{i_2}}}\cdots\overline{\frac{\partial F_k}{\partial x_{i_k}}}\\
&+ \bar H \sum_{1\le i_1,\dots,i_k\le n} \bar P[\bar x_{i_1},\dots,\bar x_{i_k}]\,\overline{\frac{\partial G}{\partial x_{i_1}}}\,\overline{\frac{\partial F_2}{\partial x_{i_2}}}\cdots\overline{\frac{\partial F_k}{\partial x_{i_k}}}\\
&= \sum_{1\le i_1,\dots,i_k\le n} \bar P[\bar x_{i_1},\dots,\bar x_{i_k}]\,\overline{\frac{\partial GH}{\partial x_{i_1}}}\,\overline{\frac{\partial F_2}{\partial x_{i_2}}}\cdots\overline{\frac{\partial F_k}{\partial x_{i_k}}}\\
&= \sum_{1\le i_1,\dots,i_k\le n} \bar P[\bar x_{i_1},\dots,\bar x_{i_k}]\,\overline{\frac{\partial F_1}{\partial x_{i_1}}}\,\overline{\frac{\partial F_2}{\partial x_{i_2}}}\cdots\overline{\frac{\partial F_k}{\partial x_{i_k}}},
\end{aligned}
$$

So, we have proved Formula (4.25) for all $F_1,\dots,F_k \in \mathcal{A}$. □

Remark 4.15. As a consequence of this proposition, one can see that any skew-symmetric k-derivation of $\bar{\mathcal{A}}$ comes from a skew-symmetric k-derivation of $\mathcal{A}$. Let us precise this point. To do that, let us denote by $\mathrm{Hom}_{\mathcal{I}}(\wedge^\bullet\mathcal{A},\mathcal{A})$ the space of all map $P\in\mathrm{Hom}(\wedge^\bullet\mathcal{A},\mathcal{A})$, satifying $P(F_1,\dots,F_k)\in\mathcal{I}$, if there exists $1\le l\le k$, such that $F_l\in\mathcal{I}$. Now, let us point out that the projection map $\wp$ extends naturally to a map (also denoted by $\wp$)

$$
\begin{array}{rcl}
\wp : \mathrm{Hom}_{\mathcal{I}}(\wedge^\bullet\mathcal{A},\mathcal{A}) & \mapsto & \mathrm{Hom}(\wedge^\bullet\bar{\mathcal{A}},\bar{\mathcal{A}}),\\
P & \to & \wp_* P
\end{array}
$$

given, for $k\in\mathbf{N}$ and $P\in\mathrm{Hom}_{\mathcal{I}}(\wedge^k\mathcal{A},\mathcal{A})$ by:

$$\wp_*P(\bar F_1,\dots,\bar F_k) := \wp\Big(P(F_1,\dots,F_k)\Big),$$

for all $F_1,\dots,F_k\in\mathcal{A}$. Moreover, if $P\in\mathrm{Hom}_{\mathcal{I}}(\wedge^\bullet\mathcal{A},\mathcal{A})$ is a skew-symmetric k-derivation of $\mathcal{A}$ ($P\in\mathfrak{X}^k(\mathcal{A})$), then, $\wp_*P\in\mathfrak{X}^k(\bar{\mathcal{A}})$ is a skew-symmetric k-derivation of $\bar{\mathcal{A}}$. The previous Proposition 4.14 says that for any $\bar P\in\mathfrak{X}^k(\bar{\mathcal{A}})$, skew-symmetric k-derivation of $\bar{\mathcal{A}}$, we have:

$$\bar P = \wp_* P, \tag{4.26}$$

where $P\in\mathfrak{X}^k(\mathcal{A})$ is the skew-symmetric k-derivation of $\mathcal{A}$, defined, for $F_1,\dots,F_k\in\mathcal{A}$, by:

$$P[F_1,\dots,F_k] := \sum_{1\le i_1\le\dots\le i_k\le n} P_{i_1,\dots,i_k}\frac{\partial F_1}{\partial x_{i_1}}\cdots\frac{\partial F_k}{\partial x_{i_k}},$$

with $P_{i_1,\dots,i_k}\in\mathcal{A}$, a representant of $\bar P[\bar x_{i_1},\dots,\bar x_{i_k}]\in\bar{\mathcal{A}}$. (With the help of Proposition 4.14, one can easily verify that $P\in\mathfrak{X}^k(\mathcal{A})$ and that $P\in\mathrm{Hom}_{\mathcal{I}}(\wedge^k\mathcal{A},\mathcal{A})$.)

Proposition 4.14 implies immediately the following

Corollary 4.16. *Let $\bar{\mathcal{A}}$ be the algebra of regular functions over an affine variety $M\subseteq\mathbf{F}^n$. Then, a skew-symmetric k-derivation of $\bar{\mathcal{A}}$ is totally determined by its values on the generators of the algebra $\bar{\mathcal{A}}$. Equivalently, if such a skew-symmetric k-derivation of $\bar{\mathcal{A}}$ vanishes on the generators, then, it is equal to zero.*

Poisson structure and skew-symmetric multi-derivations of $\mathcal{F}_\varphi$

Now, we will apply what we have seen in the last paragraph to the case that interests us: the surface $\mathcal{F}_\varphi$ in $\mathbf{F}^3$, given by a weight homogeneous polynomial φ, that admits an isolated singularity at the origin. We adapt the notations of the previous paragraph to our context. Now, $\mathcal{A}$ will denote the polynomial algebra $\mathcal{A} := \mathbf{F}[x,y,z]$. Let $\varphi \in \mathcal{A}$ be a weight homogeneous polynomial with an isolated singularity. We consider the affine variety $\mathcal{F}_\varphi : \{\varphi = 0\} \subset \mathbf{F}^3$ and its algebra of regular functions

$$\mathcal{A}_\varphi := \frac{\mathbf{F}[x,y,z]}{\langle \varphi \rangle}.$$

So that, under the notations of the last paragraph, $\mathcal{I}$ is the ideal $\langle \varphi \rangle$ and $\bar{A}$ is now $\mathcal{A}_\varphi$.

Because φ is a Casimir, $\langle \varphi \rangle$ is a Poisson ideal for $(\mathcal{A}, \{\cdot\,,\cdot\}_\varphi)$ and the Poisson structure $\{\cdot\,,\cdot\}_\varphi$ restricts naturally to $\mathcal{F}_\varphi$, that is to say goes down to the quotient $\mathcal{A}_\varphi$. That leads to a Poisson structure on $\mathcal{A}_\varphi$, denoted by $\{\cdot\,,\cdot\}_{\mathcal{A}_\varphi}$.

Let us use the same notations than above, where $\mathcal{I}$ is now the ideal $\langle \varphi \rangle$. We have the natural projection map

$$\begin{aligned} \wp : \mathcal{A} &\to \mathcal{A}_\varphi \\ F \ &\mapsto \wp(F) = \bar{F}, \end{aligned}$$

and, for each $F, G \in \mathcal{A}$, we have $\{\wp(F), \wp(G)\}_{\mathcal{A}_\varphi} = \wp\left(\{F, G\}_\varphi\right)$ (that is to say, $\wp$ is a Poisson morphism between $\mathcal{A}$ and $\mathcal{A}_\varphi$).

In the language of Remark 4.15, the fact that φ is a Casimir of $\{\cdot\,,\cdot\}_\varphi$ implies that $\{\cdot\,,\cdot\}_\varphi \in \mathrm{Hom}_{\langle\varphi\rangle}(\wedge^\bullet \mathcal{A}, \mathcal{A})$. Thus, according to Remark 4.15, $\wp_* \{\cdot\,,\cdot\}_\varphi$ is well-defined and is an element of $\mathfrak{X}^2(\mathcal{A}_\varphi)$, which we denote by $\{\cdot\,,\cdot\}_{\mathcal{A}_\varphi}$. The Jacobi identity is satisfied by $\{\cdot\,,\cdot\}_{\mathcal{A}_\varphi}$ because it is by $\{\cdot\,,\cdot\}_\varphi$.

In the following proposition, we give the Poisson cohomology spaces of the Poisson algebra $(\mathcal{A}_\varphi, \{\cdot\,,\cdot\}_{\mathcal{A}_\varphi})$. That leads to consider the skew-symmetric multi-derivations of the algebra $\mathcal{A}_\varphi$ and the Poisson coboundary operators, associated to $\{\cdot\,,\cdot\}_{\mathcal{A}_\varphi}$. By a slight abuse of notations we will, for an element $\vec{F} = (F_1, F_2, F_3) \in \mathcal{A}^3$, denote by $\wp(\vec{F}\,)$, the element $(\wp(F_1), \wp(F_2), \wp(F_3)) \in \mathcal{A}_\varphi^3$.

Proposition 4.17. *If $\varphi \in \mathcal{A}$ is weight homogeneous with an isolated singularity, the Poisson cohomology spaces of the algebra $(\mathcal{A}_\varphi, \{\cdot\,,\cdot\}_{\mathcal{A}_\varphi})$, denoted by $H^k(\mathcal{A}_\varphi)$, are given by:*

$$\mathrm{Cas}(\mathcal{A}_\varphi) = H^0(\mathcal{A}_\varphi) \simeq \left\{ \wp(F) \in \mathcal{A}_\varphi \mid \vec{\nabla} F \times \vec{\nabla} \varphi \in \langle \varphi \rangle \right\},$$

$$H^1(\mathcal{A}_\varphi) \simeq \frac{\left\{ \wp\Big(\vec{F}\Big) \in \mathcal{A}_\varphi^3 \mid \vec{F} \cdot \vec{\nabla}\varphi \in \langle\varphi\rangle,\ -\vec{\nabla}(\vec{F} \cdot \vec{\nabla}\varphi) + \mathrm{Div}(\vec{F})\, \vec{\nabla}\varphi \in \langle \varphi \rangle \right\}}{\left\{ \wp\Big(\vec{\nabla} F \times \vec{\nabla}\varphi\Big) \mid F \in \mathcal{A} \right\}},$$

$$H^2(\mathcal{A}_\varphi) \simeq \frac{\left\{\wp\Big(\vec{F}\Big) \in \mathcal{A}_\varphi^3 \mid \vec{F} \times \vec{\nabla}\varphi \in \langle\varphi\rangle\right\}}{\left\{\wp\Big(-\vec{\nabla}(\vec{F}\cdot\vec{\nabla}\varphi) + \mathrm{Div}(\vec{F})\,\vec{\nabla}\varphi\Big) \mid \vec{F} \in \mathcal{A}^3; \vec{F}\cdot\vec{\nabla}\varphi \in \langle\varphi\rangle\right\}},$$

and $H^3(\mathcal{A}_\varphi) \simeq \{0\}$.

Subsequently, we denote by $Z^k(\mathcal{A}_\varphi)$ (respectively $B^k(\mathcal{A}_\varphi)$) the space of all k-cocycles (respectively k-coboundaries) of $\mathcal{A}_\varphi$.

Proof. We first have to determine the skew-symmetric multi-derivations of $\mathcal{A}_\varphi$. To do this, we use Proposition 4.14, in the particular case where $n = 3$, $\mathcal{I} = \langle\varphi\rangle$ and $\bar{\mathcal{A}} = \mathcal{A}_\varphi$. Let us first consider the derivations of $\mathcal{A}_\varphi$. Let $\bar{\mathcal{V}} \in \mathfrak{X}^1(\mathcal{A}_\varphi)$ be a derivation of $\mathcal{A}_\varphi$ and let $\vec{F} = (F_1, F_2, F_3) \in \mathcal{A}^3$ be a triplet of polynomials such that:

$$\overline{\mathcal{V}}[\bar{x}] = \bar{F}_1, \quad \overline{\mathcal{V}}[\bar{y}] = \bar{F}_2, \quad \overline{\mathcal{V}}[\bar{z}] = \bar{F}_3.$$

According to Proposition 4.14 and especially Remark 4.15, $\overline{\mathcal{V}} \in \mathfrak{X}^1(\mathcal{A}_\varphi)$ is given by the formula:

$$\overline{\mathcal{V}} = \wp_*\mathcal{V}$$

where $\mathcal{V} \in \mathfrak{X}^1(\mathcal{A})$ is the derivation of $\mathcal{A}$, given by:

$$\mathcal{V} = F_1\frac{\partial}{\partial x} + F_2\frac{\partial}{\partial y} + F_3\frac{\partial}{\partial z}.$$

Moreover, as a well-defined linear map on $\mathcal{A}_\varphi$, $\overline{\mathcal{V}}$ should satisfy the following condition:

$$\overline{\mathcal{V}}[\bar{0}] = \overline{\mathcal{V}[\varphi]} = \bar{F}_1\overline{\frac{\partial\varphi}{\partial x}} + \bar{F}_2\overline{\frac{\partial\varphi}{\partial y}} + \bar{F}_3\overline{\frac{\partial\varphi}{\partial z}} = \bar{0},$$

i.e.,

$$\mathcal{V}[\varphi] = F_1\frac{\partial\varphi}{\partial x} + F_2\frac{\partial\varphi}{\partial y} + F_3\frac{\partial\varphi}{\partial z} = \vec{F}\cdot\vec{\nabla}\varphi \in \langle\varphi\rangle.$$

Conversely, for any derivation $\mathcal{V} \in \mathfrak{X}^1(\mathcal{A})$ of $\mathcal{A}$, satisfying $\mathcal{V}[\varphi] \in \langle\varphi\rangle$, the formula:

$$\overline{\mathcal{V}}[\bar{F}] = \overline{\mathcal{V}[F]}, \text{ for all } F \in \mathcal{A},$$

equivalent to:

$$\overline{\mathcal{V}} = \wp_*\mathcal{V},$$

defines a derivation $\overline{\mathcal{V}}$ of $\bar{\mathcal{A}}$. Thus,

$$\mathfrak{X}^1(\mathcal{A}_\varphi) \simeq \{\wp(\vec{F}) \in \mathcal{A}_\varphi^3 \mid \vec{F}\cdot\vec{\nabla}\varphi \in \langle\varphi\rangle\}. \tag{4.27}$$

With the same reasoning, we obtain

$$\mathfrak{X}^2(\mathcal{A}_\varphi) \simeq \{\wp(\vec{F}) \in \mathcal{A}_\varphi^3 \mid \vec{F}\times\vec{\nabla}\varphi \in \langle\varphi\rangle\}. \tag{4.28}$$

It is clear that $\mathfrak{X}^0(\mathcal{A}_\varphi) \simeq \mathcal{A}_\varphi$ and $\mathfrak{X}^k(\mathcal{A}_\varphi) \simeq \{0\}$, for $k \geq 4$, let us now consider the space $\mathfrak{X}^3(\mathcal{A}_\varphi)$.

In the same way than above, we get $\mathfrak{X}^3(\mathcal{A}_\varphi) = \{\wp(F) \in \mathcal{A}_\varphi \mid F\vec{\nabla}\varphi \in \langle\varphi\rangle\}$. However, if $F \in \mathcal{A}$ satisfies $F\vec{\nabla}\varphi = \varphi\vec{G}$, with $\vec{G} \in \mathcal{A}^3$, then we have $\vec{G} \times \vec{\nabla}\varphi = \vec{0}$ and Proposition 3.6 (and the exactness of the Koszul complex associated to φ) implies the existence of an element $H \in \mathcal{A}$ satifying $\vec{G} = H\vec{\nabla}\varphi$, so that $F = H\varphi \in \langle\varphi\rangle$. That leads to

$$\mathfrak{X}^3(\mathcal{A}_\varphi) \simeq \{0\}. \tag{4.29}$$

Now, let us consider the Poisson coboundary operators of the Poisson algebra $(\mathcal{A}_\varphi, \{\cdot\,,\cdot\}_{\mathcal{A}_\varphi})$, denoted by $\delta^k_{\mathcal{A}_\varphi}$. Let $\overline{P} \in \mathfrak{X}^k(\mathcal{A}_\varphi)$ be a skew-symmetry k-derivation of $\mathcal{A}_\varphi$. According to Proposition 4.14 and Remark 4.15, there exists $P \in \mathfrak{X}^k(\mathcal{A})$, such that $\overline{P} = \wp_* P$. Using the definition of $\delta^k_{\mathcal{A}_\varphi}$ and the fact that $\wp$ is a Poisson morphism between $(\mathcal{A}, \{\cdot\,,\cdot\}_\varphi)$ and $(\mathcal{A}_\varphi, \{\cdot\,,\cdot\}_{\mathcal{A}_\varphi})$, we obtain:

$$\delta^k_{\mathcal{A}_\varphi}(\,\overline{P}\,) = \delta^k_{\mathcal{A}_\varphi}(\wp_* P) = \wp_*\,\delta^k_\varphi(P),$$

that is to say, for all $F_0, \ldots, F_k \in \mathcal{A}$,

$$\delta^k_{\mathcal{A}_\varphi}(\,\overline{P}\,)[\bar{F}_0, \ldots, \bar{F}_k] = \wp\left(\delta^k_\varphi(P)[F_0, \ldots, F_k]\right).$$

Moreover, if, under the isomorphisms (4.27) and (4.28), $\overline{P}$ is defined by an element $\wp(F) = \bar{F} \in \mathcal{A}_\varphi$, respectively by a triplet $\wp(\vec{F}) \in \mathcal{A}^3_\varphi$, then Proposition 4.14 allows us to chose P, defined by $F \in \mathcal{A}$, respectively by $\vec{F} \in \mathcal{A}^3$, under the correspondences given in the paragraph 3.1.1. That leads to:

$$\begin{aligned}
&\delta^0_{\mathcal{A}_\varphi}(\wp(F)) = \wp\left(\vec{\nabla}F \times \vec{\nabla}\varphi\right), \quad \text{for } \wp(F) \in \mathcal{A}_\varphi \simeq \mathfrak{X}^0(\mathcal{A}_\varphi),\\
&\delta^1_{\mathcal{A}_\varphi}(\wp(\vec{F})) = \wp\left(-\vec{\nabla}(\vec{F}\cdot\vec{\nabla}\varphi) + \mathrm{Div}(\vec{F})\vec{\nabla}\varphi\right),\\
&\qquad\qquad\qquad \text{for } \wp(\vec{F}) \in \{\wp(\vec{G}) \in \mathcal{A}^3_\varphi \mid \vec{G}\cdot\vec{\nabla}\varphi \in \langle\varphi\rangle\} \simeq \mathfrak{X}^1(\mathcal{A}_\varphi),\\
&\delta^2_{\mathcal{A}_\varphi}(\wp(\vec{F})) = 0, \quad \text{for } \wp(\vec{F}) \in \{\wp(\vec{G}) \in \mathcal{A}^3_\varphi \mid \vec{G}\times\vec{\nabla}\varphi \in \langle\varphi\rangle\} \simeq \mathfrak{X}^2(\mathcal{A}_\varphi),
\end{aligned}$$

while the writing of the Poisson cohomology spaces follows. □

Remark 4.18. To prove that $\mathfrak{X}^3(\mathcal{A}_\varphi) \simeq \{0\}$, we only used the exactness of the second part of the Koszul complex (3.11) which remains true even if φ is a square-free weight homogeneous polynomial, not necessarily with an isolated singularity (See Remark 3.8). So that, if $\varphi \in \mathcal{A}$ is a square-free weight homogeneous polynomial, we still have:

$$\mathfrak{X}^3(\mathcal{A}_\varphi) \simeq \{0\},$$

and

$$H^3(\mathcal{A}_\varphi) \simeq \{0\}.$$

We point out that if the condition "square-free" is not satisfied by φ, then, it is not true in general that the unique skew-symmetry 3-derivation on $\mathcal{A}_\varphi$ is zero. For example, let us consider the homogeneous polynomial $x^2 \in \mathcal{A}$. Then, $x\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ is an non zero element of $\mathfrak{X}^3(\mathcal{A}_{x^2})$.

4.3.2 The space $H^0(\mathcal{A}_\varphi)$

In this section, we still consider $\varphi \in \mathcal{A}$, a weight homogeneous polynomial, with an isolated singularity, and the Poisson bracket on $\mathcal{A}_\varphi$, denoted by $\{\cdot\,,\cdot\}_{\mathcal{A}_\varphi}$. We describe the zeroth Poisson cohomology space, that is to say the space of the Casimirs of $(\mathcal{A}_\varphi, \{\cdot\,,\cdot\}_{\mathcal{A}_\varphi})$ in the following proposition.

Proposition 4.19. *If $\varphi \in \mathcal{A} = \mathbf{F}[x, y, z]$ is weight homogeneous with an isolated singularity, the zeroth Poisson cohomology space of the singular surface defined by this polynomial is given by*

$$H^0(\mathcal{A}_\varphi) = \mathrm{Cas}(\mathcal{A}_\varphi) \simeq \mathbf{F}.$$

Proof. Let $F \in \mathcal{A}$ be a weight homogeneous polynomial such that $\wp(F) \in H^0(\mathcal{A}_\varphi)$. Then $\vec{\nabla} F \times \vec{\nabla}\varphi \in \langle\varphi\rangle$ i.e., there exists $\vec{G} \in \mathcal{A}^3$ satifying $\vec{\nabla} F \times \vec{\nabla}\varphi = \varphi\, \vec{G}$. It follows that $\vec{G} \cdot \vec{\nabla}\varphi = 0$ and Proposition 3.6 (the exactness of the Koszul complex associated to φ) implies the existence of an element $\vec{H} \in \mathcal{A}^3$ such that $\vec{G} = \vec{H} \times \vec{\nabla}\varphi$. Summing up, $(\vec{\nabla} F - \varphi\vec{H}) \times \vec{\nabla}\varphi = \vec{0}$, and we can apply Proposition 3.6 again to obtain a $K \in \mathcal{A}$ satifying $\vec{\nabla} F = \varphi\vec{H} + K\, \vec{\nabla}\varphi$. Euler's Formula (3.5) gives

$$\varpi(F)\, F = \vec{\nabla} F \cdot \vec{e}_\varpi = \varphi(\vec{H} \cdot \vec{e}_\varpi + \varpi(\varphi)\, K).$$

So, $F \in \langle\varphi\rangle$ unless $\varpi(F)$, the (weighted) degree of F, is zero, thus $H^0(\mathcal{A}_\varphi) \simeq \mathbf{F}$. □

4.3.3 The space $H^1(\mathcal{A}_\varphi)$

This section is devoted to the determination of the first Poisson cohomology space of $(\mathcal{A}_\varphi, \{\cdot\,,\cdot\}_{\mathcal{A}_\varphi})$, where $\varphi \in \mathcal{A} = \mathbf{F}[x, y, z]$ is weight homogeneous with an isolated singularity.

Remark 4.20. Using Proposition 4.19, we can simplify the writing of $Z^1(\mathcal{A}_\varphi)$. Let indeed $\vec{F} \in \mathcal{A}^3$ be an element satisfying: $-\vec{\nabla}(\vec{F} \cdot \vec{\nabla}\varphi) + \mathrm{Div}(\vec{F})\, \vec{\nabla}\varphi \in \langle\varphi\rangle$. Then $-\vec{\nabla}(\vec{F} \cdot \vec{\nabla}\varphi) \times \vec{\nabla}\varphi \in \langle\varphi\rangle$, that is to say $\wp(\vec{F} \cdot \vec{\nabla}\varphi) \in H^0(\mathcal{A}_\varphi) \simeq \mathbf{F}$, according to Proposition 4.19. For degree reasons, this leads to $\vec{F} \cdot \vec{\nabla}\varphi \in \langle\varphi\rangle$. So, we can simply write

$$Z^1(\mathcal{A}_\varphi) = \left\{\wp(\vec{F}) \in \mathcal{A}_\varphi^3 \mid -\vec{\nabla}(\vec{F} \cdot \vec{\nabla}\varphi) + \mathrm{Div}(\vec{F})\, \vec{\nabla}\varphi \in \langle\varphi\rangle\right\}$$

Now, let us give the main result of this section (we recall that $|\varpi|$ is the sum of the weights $\varpi_1, \varpi_2, \varpi_3$ of the variables x, y, z and that the family $\{u_j\}$ is a weight-homogeneous $\mathbf{F}$-basis of $\mathcal{A}_{sing}$ and is defined in Section 3.2.4).

Proposition 4.21. *If $\varphi \in \mathcal{A} = \mathbf{F}[x,y,z]$ is weight homogeneous with an isolated singularity, then the first Poisson cohomology space of the singular surface $\mathcal{F}_\varphi$: $\{\varphi = 0\}$ is given by*

$$H^1(\mathcal{A}_\varphi) \simeq \bigoplus_{\substack{j=0 \\ \varpi(u_j)=\varpi(\varphi)-|\varpi|}}^{\mu-1} \mathbf{F}\wp(u_j\,\vec{e}_\varpi).$$

In particular, if $\varpi(\varphi) < |\varpi|$ then $H^1(\mathcal{A}_\varphi) \simeq \{0\}$.

Proof. Let $\vec{F} \in \mathcal{A}^3$ satisfy $\wp\left(\vec{F}\right) \in Z^1(\mathcal{A}_\varphi)$, it means that there exists $\vec{K} \in \mathcal{A}^3$ satisfying $\delta^1_\varphi(\vec{F}) = \varphi\vec{K}$. Then $0 = \delta^2_\varphi(\varphi\vec{K}) = \varphi\,\delta^2_\varphi(\vec{K})$, because, as we said in Remark 2.17, the operator δ^2_φ commutes with the multiplication by φ. So $\varphi\,\vec{K} \in B^2(\mathcal{A},\varphi)$ and $\vec{K} \in Z^2(\mathcal{A},\varphi)$. According to Proposition 3.19,

$$\vec{K} \in B^2(\mathcal{A},\varphi) \oplus \bigoplus_{\substack{j=1 \\ \varpi(u_j)\neq\varpi(\varphi)-|\varpi|}}^{\mu-1} \mathrm{Cas}(\mathcal{A},\varphi)\vec{\nabla}u_j$$
$$\oplus \bigoplus_{\substack{k=0 \\ \varpi(u_k)=\varpi(\varphi)-|\varpi|}}^{\mu-1} \mathrm{Cas}(\mathcal{A},\varphi)u_k\vec{\nabla}\varphi \oplus \bigoplus_{\substack{l=1 \\ \varpi(u_l)=\varpi(\varphi)-|\varpi|}}^{\mu-1} \mathbf{F}\vec{\nabla}u_l.$$

Each of the first three summands is stable by multiplication by φ, while Remark 3.21 gives

$$\bigoplus_{\substack{l=1 \\ \varpi(u_l)=\varpi(\varphi)-|\varpi|}}^{\mu-1} \varphi\mathbf{F}\vec{\nabla}u_l \subset B^2(\mathcal{A},\varphi).$$

As a consequence, since $\varphi\vec{K} \in B^2(\mathcal{A},\varphi)$,

$$\vec{K} \in B^2(\mathcal{A},\varphi) \oplus \bigoplus_{\substack{l=1 \\ \varpi(u_l)=\varpi(\varphi)-|\varpi|}}^{\mu-1} \mathbf{F}\vec{\nabla}u_l.$$

So there exist $\vec{H} \in \mathcal{A}^3$ and elements $\lambda_l \in \mathbf{F}$, with l satisfying $\varpi(u_l) = \varpi(\varphi) - |\varpi|$, such that

$$\vec{K} = \delta^1_\varphi(\vec{H}) + \sum_{\substack{l=1 \\ \varpi(u_l)=\varpi(\varphi)-|\varpi|}}^{\mu-1} \lambda_l\vec{\nabla}u_l.$$

For all $1 \leq l \leq \mu - 1$ such that $\varpi(u_l) = \varpi(\varphi) - |\varpi|$, we have $\varphi\vec{\nabla}u_l = -\delta^1_\varphi\left(\frac{1}{\varpi(\varphi)}u_l\,\vec{e}_\varpi\right)$, according to Formula (3.23), so that

$$\delta^1_\varphi(\vec{F}) = \varphi\vec{K} = \delta^1_\varphi\left(\varphi\vec{H} - \sum_{\substack{l=1\\ \varpi(u_l)=\varpi(\varphi)-|\varpi|}}^{\mu-1} \frac{\lambda_l}{\varpi(\varphi)}u_l\,\vec{e}_\varpi\right).$$

This implies

$$\vec{F} - \varphi\vec{H} + \sum_{\substack{l=1\\ \varpi(u_l)=\varpi(\varphi)-|\varpi|}}^{\mu-1} \frac{\lambda_l}{\varpi(\varphi)}u_l\,\vec{e}_\varpi \in Z^1(\mathcal{A},\varphi). \tag{4.30}$$

• If $\varpi(\varphi) \neq |\varpi|$, then Proposition 3.14 implies that (4.30) belongs to $B^1(\mathcal{A},\varphi)$, so that

$$\wp(\vec{F}) \in \sum_{\substack{l=1\\ \varpi(u_l)=\varpi(\varphi)-|\varpi|}}^{\mu-1} \mathbf{F}\wp(u_l\,\vec{e}_\varpi) + B^1(\mathcal{A}_\varphi).$$

• If $\varpi(\varphi) = |\varpi|$ then (4.30) is simply the equation $\vec{F} - \varphi\vec{H} \in Z^1(\mathcal{A},\varphi) \simeq B^1(\mathcal{A},\varphi) + \mathrm{Cas}(\mathcal{A},\varphi)\,\vec{e}_\varpi$, according to Proposition 3.14. So, we have $\wp(\vec{F}) \in \mathbf{F}\,\wp(\vec{e}_\varpi) + B^1(\mathcal{A}_\varphi)$. As we have $\varpi(u_l) \geq 1$, if $1 \leq l \leq \mu - 1$, the result of both cases can be summarized as follows:

$$Z^1(\mathcal{A}_\varphi) \subseteq B^1(\mathcal{A}_\varphi) + \sum_{\substack{l=0\\ \varpi(u_l)=\varpi(\varphi)-|\varpi|}}^{\mu-1} \mathbf{F}\wp(u_l\,\vec{e}_\varpi).$$

Euler's Formula (3.5) (or Formula (3.23)) implies that $\wp(u_l\,\vec{e}_\varpi) \in Z^1(\mathcal{A}_\varphi)$ ($\delta^1_\varphi(u_l\,\vec{e}_\varpi) \in \langle\varphi\rangle$), when $\varpi(u_l) = \varpi(\varphi) - |\varpi|$, so that the other inclusion holds too. It also allows us to show that the above sum is a direct one. Hence the result about $H^1(\mathcal{A}_\varphi)$. □

4.3.4 The space $H^2(\mathcal{A}_\varphi)$

We now compute the second Poisson cohomology space of $(\mathcal{A}_\varphi, \{\cdot\,,\cdot\}_{\mathcal{A}_\varphi})$, where $\varphi \in \mathcal{A} = \mathbf{F}[x,y,z]$ is weight homogeneous with an isolated singularity. We recall that $\wp$ is the natural projection map, $\wp : \mathcal{A} \to \mathcal{A}_\varphi = \mathbf{F}[x,y,z]/\langle\varphi\rangle$.

Proposition 4.22. *If $\varphi \in \mathcal{A} = \mathbf{F}[x,y,z]$ is weight homogeneous with an isolated singularity, then $H^2(\mathcal{A}_\varphi)$ is given by*

$$H^2(\mathcal{A}_\varphi) \simeq \bigoplus_{\substack{j=0\\ \varpi(u_j)=\varpi(\varphi)-|\varpi|}}^{\mu-1} \mathbf{F}\wp(u_j\vec{\nabla}\varphi).$$

Remark 4.23. It follows from Propositions 4.21 and 4.22 that there is a natural isomorphism between $H^1(\mathcal{A}_\varphi)$ and $H^2(\mathcal{A}_\varphi)$, that maps the element $u_j\,\vec{e}_\varpi$ (with $\varpi(u_j) = \varpi(\varphi) - |\varpi|$) to the element $u_j\,\vec{\nabla}\varphi$ of $H^2(\mathcal{A}_\varphi)$.

Proof. First, we show that the family $\left\{\wp(u_j\vec{\nabla}\varphi) \mid \varpi(u_j) = \varpi(\varphi) - |\varpi|\right\}$ generates the $\mathbf{F}$-vector space $H^2(\mathcal{A}_\varphi)$. Let $\vec{H} \in \mathcal{A}^3$ such that $\wp(\vec{H}) \in Z^2(\mathcal{A}_\varphi)$, that is to say, such that there exists $\vec{G} \in \mathcal{A}^3$ satisfying $\vec{H} \times \vec{\nabla}\varphi = \varphi\,\vec{G}$. According to Remark 3.7, we may suppose $\vec{H} \in \mathfrak{X}^2(\mathcal{A})_d$ and $\vec{G} \in \mathfrak{X}^1(\mathcal{A})_d$, with $d \in \mathbf{Z}$. Since $\vec{G} \cdot \vec{\nabla}\varphi = 0$, Proposition 3.6 implies that $\vec{G} = \vec{K} \times \vec{\nabla}\varphi$ and $\vec{H} = \varphi\vec{K} + F\vec{\nabla}\varphi$, with $F \in \mathfrak{X}^3(\mathcal{A})_{d-\varpi(\varphi)}$ and $\vec{K} \in \mathfrak{X}^2(\mathcal{A})_{d-\varpi(\varphi)}$.

If $d < \varpi(\varphi) - |\varpi|$ then $F = 0$ and $\vec{H} \in \langle\varphi\rangle$; otherwise $\wp(\vec{H}) = \wp(F\vec{\nabla}\varphi)$, while, using Formulas (3.5) and (3.6), we get

$$\delta^1_\varphi(F\vec{e}_\varpi) = (d - 2\varpi(\varphi) + 2|\varpi|)\,F\vec{\nabla}\varphi - \varpi(\varphi)\,\varphi\vec{\nabla}F.$$

That leads, in the case $d \neq 2(\varpi(\varphi) - |\varpi|)$, to $\wp(\vec{H}) = \wp(F\vec{\nabla}\varphi) \in B^2(\mathcal{A}_\varphi)$. Therefore, let us suppose that $d = 2(\varpi(\varphi) - |\varpi|)$, so that $\varpi(F) = \varpi(\varphi) - |\varpi|$. For degree reasons, the projection map $\mathcal{A} \to \mathcal{A}_{sing} = \mathcal{A}/\langle\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\rangle$ restricts to an injective map $\mathcal{A}_{\varpi(\varphi)-|\varpi|} \to \mathcal{A}_{sing}$, so that F is a $\mathbf{F}$-linear combination of the u_j satisfying $\varpi(u_j) = \varpi(\varphi) - |\varpi|$, that leads to

$$\wp(\vec{H}) \in \sum_{\substack{j=0 \\ \varpi(u_j)=\varpi(\varphi)-|\varpi|}}^{\mu-1} \mathbf{F}\wp(u_j\vec{\nabla}\varphi),$$

and for all j, $u_j\,\vec{\nabla}\varphi \in Z^2(\mathcal{A}_\varphi)$.

It suffices now to show that this family is $\mathbf{F}$-free, modulo $B^2(\mathcal{A}_\varphi)$. It is empty if $\varpi(\varphi) < |\varpi|$, so we suppose $\varpi(\varphi) \geq |\varpi|$. Let λ_j be elements of $\mathbf{F}$ with j such that $\varpi(u_j) = \varpi(\varphi) - |\varpi|$ and let $\vec{L}, \vec{J} \in \mathcal{A}^3$ satisfying

$$\begin{aligned} \sum_{\substack{j=0 \\ \varpi(u_j)=\varpi(\varphi)-|\varpi|}}^{\mu-1} \lambda_j u_j\vec{\nabla}\varphi &= -\vec{\nabla}(\vec{L}\cdot\vec{\nabla}\varphi) + \mathrm{Div}(\vec{L})\vec{\nabla}\varphi + \varphi\vec{J} \\ &= \delta^1_\varphi(\vec{L}) + \varphi\vec{J}, \end{aligned} \tag{4.31}$$

where the right hand side is an arbitrary representative of an element of $B^2(\mathcal{A}_\varphi)$. As the left hand side belongs to the space $\mathfrak{X}^2(\mathcal{A})_{2\varpi(\varphi)-2|\varpi|}$, we may suppose that $\vec{L} \in \mathfrak{X}^1(\mathcal{A})_{\varpi(\varphi)-|\varpi|}$ and $\vec{J} \in \mathfrak{X}^2(\mathcal{A})_{\varpi(\varphi)-2|\varpi|}$.

The equation (4.31) implies $\vec{\nabla}(\vec{L}\cdot\vec{\nabla}\varphi) \times \vec{\nabla}\varphi \in \langle\varphi\rangle$, so that $\wp\Big(\vec{L}\cdot\vec{\nabla}\varphi\Big) \in \mathrm{Cas}(\mathcal{A}_\varphi)$. For degree reasons, Proposition 4.19 leads to the existence of $G \in \mathcal{A}$ of degree $\varpi(\varphi) - |\varpi|$ such that $\vec{L}\cdot\vec{\nabla}\varphi = \varphi\,G = (G\vec{e}_\varpi \cdot \vec{\nabla}\varphi)/\varpi(\varphi)$. Then

Proposition 3.6 and the exactness of the Koszul complex imply that $\varpi(\varphi)\,\vec{L} = G\vec{e}_\varpi$ and $\delta^1_\varphi(\vec{L}) = -\varphi\vec{\nabla}G$, so that

$$\sum_{\substack{j=0 \\ \varpi(u_j)=\varpi(\varphi)-|\varpi|}}^{\mu-1} \lambda_j u_j \vec{\nabla}\varphi = -\varphi\vec{\nabla}G + \varphi\vec{J} = \varphi\vec{F}, \tag{4.32}$$

where $\vec{F} = -\vec{\nabla}G + \vec{J} \in \mathfrak{X}^2(\mathcal{A})_{\varpi(\varphi)-2|\varpi|}$. We get $\vec{F} \times \vec{\nabla}\varphi = \vec{0}$, but for degree reasons, Proposition 3.6 leads to $\vec{F} = \vec{0}$ so that, for all j, $\lambda_j = 0$, since the family $\{u_j\}$ if $\mathbf{F}$-free in $\mathcal{A}$. Hence the result. $\square$

4.4 Homology of the singular surface $\mathcal{F}_\varphi$

4.4.1 The Poisson homology complex of $\mathcal{F}_\varphi$

Now, let us determine the Poisson homology complex of the singular surface $\mathcal{F}_\varphi$. For the quotient algebra $\mathcal{A}_\varphi = \mathbf{F}[x, y, z]/\langle\varphi\rangle$, the space $\Omega^\bullet(\mathcal{A}_\varphi)$ is obtained by subjecting the $\mathcal{A}_\varphi$-module generated by the wedge products of $\mathsf{d}x, \mathsf{d}y, \mathsf{d}z$ to the relations $\varphi = 0$, $\mathsf{d}\varphi = 0$ and $\mathsf{d}\varphi \wedge \mathsf{d}x = 0$, etc. We recall the natural surjective map $\wp : \mathcal{A} \to \mathcal{A}_\varphi$, which is a Poisson morphism. This map induces another surjective map $\wp^\sharp : \Omega^k(\mathcal{A}) \to \Omega^k(\mathcal{A}_\varphi)$ between the spaces of all k-chains, which allows us to see the differential k-forms of $\mathcal{A}_\varphi$ as images of differential k-forms of $\mathcal{A}$. Thus, as the differential forms of $\mathcal{A}$ are identified to elements of $\mathcal{A}$ or $\mathcal{A}^3$, as can be seen in (3.31), we can write the spaces of all differential k-forms of $\mathcal{A}_\varphi$ as quotients of $\mathcal{A}_\varphi$ and $\mathcal{A}^3_\varphi$ and then as quotients of $\mathcal{A}$ and $\mathcal{A}^3$. We obtain, while $\Omega^0(\mathcal{A}_\varphi) \simeq \mathcal{A}_\varphi$,

$$\Omega^1(\mathcal{A}_\varphi) \simeq \frac{\mathcal{A}^3_\varphi}{\{F\vec{\nabla}\varphi \mid F \in \mathcal{A}\}} \simeq \frac{\mathcal{A}^3}{\{F\vec{\nabla}\varphi + \varphi\vec{G} \mid F \in \mathcal{A},\, \vec{G} \in \mathcal{A}^3\}},$$

$$\Omega^2(\mathcal{A}_\varphi) \simeq \frac{\mathcal{A}^3_\varphi}{\{\vec{\nabla}\varphi \times \vec{F} \mid \vec{F} \in \mathcal{A}^3\}} \simeq \frac{\mathcal{A}^3}{\{\vec{\nabla}\varphi \times \vec{F} + \varphi\vec{G} \mid \vec{F}, \vec{G} \in \mathcal{A}^3\}},$$

$$\Omega^3(\mathcal{A}_\varphi) \simeq \frac{\mathcal{A}_\varphi}{\{\vec{\nabla}\varphi \cdot \vec{F} \mid \vec{F} \in \mathcal{A}^3\}} \simeq \frac{\mathcal{A}}{\left\langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z} \right\rangle} = \mathcal{A}_{sing}.$$

Remark 4.24. Unlike for $\mathcal{A}$, there is no isomorphisms between the spaces of skew-symmetric multi-derivations and differential forms on $\mathcal{A}_\varphi$. For example, $\Omega^0(\mathcal{A}_\varphi) \simeq \mathcal{A}_\varphi$ while $\mathfrak{X}^3(\mathcal{A}_\varphi) \simeq \{0\}$ and $\mathfrak{X}^2(\mathcal{A}_\varphi) \subseteq \mathcal{A}^3_\varphi$. Observe also that $\Omega^3(\mathcal{A}_\varphi) \not\simeq \{0\}$, although $\mathcal{F}_\varphi$ is an affine variety of dimension two.

In view of the definition of the boundary operator in Paragraph 2.2.2, the operator δ^φ_k induces an operator $\Omega^k(\mathcal{A}_\varphi) \to \Omega^{k-1}(\mathcal{A}_\varphi)$, that is exactly the boundary operator associated to $(\mathcal{A}_\varphi, \{\cdot\,,\cdot\}_{\mathcal{A}_\varphi})$ and denoted by $\partial^{\mathcal{A}_\varphi}_k$, so that the Poisson homology spaces of $\mathcal{A}_\varphi$ are given by

$$H_0(\mathcal{A}_\varphi) \simeq \frac{\mathcal{A}}{\{\vec{\nabla}\varphi \cdot (\vec{\nabla} \times \vec{F}) + \varphi G \mid G \in \mathcal{A}, \vec{F} \in \mathcal{A}^3\}},$$

$$H_1(\mathcal{A}_\varphi) \simeq \frac{\{\vec{F} \in \mathcal{A}^3 \mid \vec{\nabla}\varphi \cdot (\vec{\nabla} \times \vec{F}) \in \langle\varphi\rangle\}}{\{-\vec{\nabla}(\vec{F} \cdot \vec{\nabla}\varphi) + \mathrm{Div}(\vec{F})\,\vec{\nabla}\varphi + G\vec{\nabla}\varphi + \varphi\vec{H} \mid G \in \mathcal{A}, \vec{F}, \vec{H} \in \mathcal{A}^3\}},$$

$$H_2(\mathcal{A}_\varphi) \simeq \frac{\{\vec{F} \in \mathcal{A}^3 \mid -\vec{\nabla}(\vec{F} \cdot \vec{\nabla}\varphi) + \mathrm{Div}(\vec{F})\,\vec{\nabla}\varphi \in \mathcal{I}_\varphi\}}{\{\vec{\nabla}\varphi \times \vec{H} + \varphi\vec{K} \mid \vec{H}, \vec{K} \in \mathcal{A}^3\}},$$

$$\text{where } \mathcal{I}_\varphi := \{F\vec{\nabla}\varphi + \varphi\vec{G} \mid F \in \mathcal{A}, \vec{G} \in \mathcal{A}^3\},$$

$$H_3(\mathcal{A}_\varphi) \simeq \mathcal{A}_{sing}.$$

Remark 4.25. In view of the writing of the Poisson homology groups of $\mathcal{A}$ and $\mathcal{A}_\varphi$, we can describe explicitly the map induced by $\wp$ between these groups. In fact, this map is exactly the reduction modulo φ between the spaces $H_k(\mathcal{A}, \varphi)$ and $H_k(\mathcal{A}_\varphi)$, for $k \neq 1$, and it is the reduction modulo $\mathcal{I}_\varphi$, for $k = 1$. This phenomenon will be illustrated in the determination of the Poisson homology groups of $\mathcal{A}_\varphi$.

4.4.2 The Poisson homology spaces of the singular surface $\mathcal{F}_\varphi$

In this Paragraph, $\varphi \in \mathcal{A} = \mathbf{F}[x, y, z]$ is still weight homogeneous with an isolated singularity and we determine the Poisson homology spaces of the Poisson surface $(\mathcal{F}_\varphi, \{\cdot\,,\cdot\}_{\mathcal{A}_\varphi})$.

Proposition 4.26. *If $\varphi \in \mathcal{A}$ is weight homogeneous with an isolated singularity, then the homology spaces of the singular surface $\mathcal{F}_\varphi$ are given by:*

$$H_0(\mathcal{A}_\varphi) \simeq \bigoplus_{j=0}^{\mu-1} \mathbf{F} u_j \simeq \mathcal{A}_{sing}, \qquad H_1(\mathcal{A}_\varphi) \simeq \bigoplus_{j=1}^{\mu-1} \mathbf{F} \vec{\nabla} u_j,$$

$$H_2(\mathcal{A}_\varphi) \simeq \bigoplus_{j=0}^{\mu-1} \mathbf{F} u_j\, \vec{e}_\varpi \simeq \mathcal{A}_{sing}.$$

Remark 4.27. The fact that $H_0(\mathcal{A}_\varphi) \simeq \mathcal{A}_{sing}$ was already proved by J. Alev and T. Lambre, with other methods, in [2]. Their result is more general as they only suppose that φ is a weight homogeneous polynomial, not necessarily with an isolated singularity.

Remark 4.28. The multiplication by $\vec{e}_\varpi$ gives a natural isomorphism between $H_0(\mathcal{A}_\varphi)$ and $H_2(\mathcal{A}_\varphi)$, while the operator of gradient $\vec{\nabla}$ gives a surjective map from $H_0(\mathcal{A}_\varphi)$ to $H_1(\mathcal{A}_\varphi)$.

Proof. 1. We first determine $H_0(\mathcal{A}_\varphi)$. According to Proposition 3.16 (i.e., the writing of $H^3(\mathcal{A}, \varphi)$), we have:

$$
\begin{aligned}
\mathcal{A} &= \{\vec{\nabla}\varphi \cdot (\vec{\nabla} \times \vec{F}) \mid \vec{F} \in \mathcal{A}^3\} + \sum_{\substack{j=0\\ i\in\mathbf{N}}}^{\mu-1} \mathbf{F}\varphi^i u_j,\\
&= \{\vec{\nabla}\varphi \cdot (\vec{\nabla} \times \vec{F}) + \varphi G \mid G \in \mathcal{A}, \vec{F} \in \mathcal{A}^3\} + \bigoplus_{j=0}^{\mu-1} \mathbf{F}\, u_j.
\end{aligned}
$$

Moreover this last sum is a direct one, as follows from the definition of the u_j (in Section 3.2.4) and the inclusion

$$
\{\vec{\nabla}\varphi \cdot (\vec{\nabla} \times \vec{F}) + \varphi G \mid G \in \mathcal{A}, \vec{F} \in \mathcal{A}^3\} \subseteq \left\langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z} \right\rangle,
$$

easily obtained with Euler's Formula (3.5). That leads to $H_0(\mathcal{A}_\varphi) \simeq \bigoplus_{j=0}^{\mu-1} \mathbf{F}\, u_j$.

2. Now, we use the result we obtained for $H^2(\mathcal{A}, \varphi)$ to determine the first Poisson homology space of $\mathcal{A}_\varphi$. Let $\vec{F} \in \mathcal{A}^3$ satisfying $\vec{\nabla}\varphi \cdot (\vec{\nabla} \times \vec{F}) \in \langle\varphi\rangle$, thus, there exists $G \in \mathcal{A}$ with $-\delta^2_\varphi(\vec{F}) = \vec{\nabla}\varphi \cdot (\vec{\nabla} \times \vec{F}) = \varphi\, G$.

According to Proposition 3.16, $G \in B^3(\mathcal{A}, \varphi) \oplus \bigoplus_{j=0}^{\mu-1} \mathrm{Cas}(\mathcal{A}, \varphi)\, u_j$. As both of the summands of this sum are stable by multiplication by φ and because $\varphi G \in B^3(\mathcal{A}, \varphi)$, we have $G \in B^3(\mathcal{A}, \varphi)$, i.e. there exists $\vec{K} \in \mathcal{A}^3$ satisfying $G = \vec{\nabla}\varphi \cdot (\vec{\nabla} \times \vec{K})$. Thus, $\vec{F} - \varphi\vec{K} \in Z^2(\mathcal{A}, \varphi)$ together with Proposition 3.19 imply that

$$
\vec{F} \in \sum_{j=1}^{\mu-1} \mathbf{F}\, \vec{\nabla} u_j + \{\delta^1_\varphi(\vec{L}) + G\vec{\nabla}\varphi + \varphi\vec{H} \mid G \in \mathcal{A}, \vec{L}, \vec{H} \in \mathcal{A}^3\},
$$

so that $\{\vec{\nabla} u_j \mid 1 \le j \le \mu - 1\}$ generates the $\mathbf{F}$-vector space $H_1(\mathcal{A}_\varphi)$ and it suffices to prove that $\vec{\nabla} u_1, \ldots, \vec{\nabla} u_{\mu-1}$ are linearly independent elements of $H_1(\mathcal{A}_\varphi)$. Assume therefore that there exist elements λ_j of $\mathbf{F}$ $(1 \le j \le \mu - 1)$, $\vec{K}, \vec{L} \in \mathcal{A}^3$ and $G \in \mathcal{A}$ such that

$$
\sum_{j=1}^{\mu-1} \lambda_j \vec{\nabla} u_j = -\vec{\nabla}(\vec{L} \cdot \vec{\nabla}\varphi) + \mathrm{Div}(\vec{L})\, \vec{\nabla}\varphi + G\vec{\nabla}\varphi + \varphi\vec{H}.
$$

Then, as the u_j are weight homogeneous, Euler's Formula (3.5) leads to

$$
\sum_{j=1}^{\mu-1} \lambda_j \varpi(u_j)\, u_j \in \langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z} \rangle
$$

and the definition of the u_j implies $\lambda_j = 0$, for $1 \le j \le \mu - 1$.

3. Finally, we compute the second Poisson homology space of $\mathcal{A}_\varphi$. Let $\vec{F} \in \mathcal{A}^3$ satisfying $\delta^1_\varphi(\vec{F}) \in \mathcal{I}_\varphi$, i.e. there exist $L \in \mathcal{A}$, $\vec{G} \in \mathcal{A}^3$ such that $\delta^1_\varphi(\vec{F}) = L\vec{\nabla}\varphi + \varphi\,\vec{G}$.

• Let us study the term $\varphi\,\vec{G}$. We first point out that $L\vec{\nabla}\varphi \in Z^2(\mathcal{A},\varphi)$, so that $\varphi\,\vec{G} = \delta^1(\vec{F}) - L\vec{\nabla}\varphi \in Z^2(\mathcal{A},\varphi)$ and $\vec{G} \in Z^2(\mathcal{A},\varphi)$. Using Proposition 3.19, Formula (3.23) and the fact that δ^1_φ commutes with φ, we obtain the existence of $\vec{H} \in \mathcal{A}^3$ and $c_j \in \mathbf{F}$, such that:

$$\begin{aligned} \varphi\,\vec{G} \in \delta^1_\varphi &\left(\varphi\,\vec{H} + \sum_{\substack{j=1 \\ \varpi(u_j)=\varpi(\varphi)-|\varpi|}}^{\mu-1} c_j u_j\,\vec{e}_\varpi \right) \\ &+ \bigoplus_{\substack{j=1 \\ \varpi(u_j)\neq\varpi(\varphi)-|\varpi|}}^{\mu-1} \operatorname{Cas}(\mathcal{A},\varphi)\vec{\nabla}u_j \oplus \bigoplus_{\substack{j=0 \\ \varpi(u_j)=\varpi(\varphi)-|\varpi|}}^{\mu-1} \operatorname{Cas}(\mathcal{A},\varphi)u_j\vec{\nabla}\varphi, \end{aligned} \tag{4.33}$$

• Next, we consider the term $L\vec{\nabla}\varphi$. According to Proposition 3.16, there exists $\vec{K} \in \mathcal{A}^3$ such that $-L \in \delta^2_\varphi(\vec{K}) + \operatorname{Cas}(\mathcal{A},\varphi)\otimes_{\mathbf{F}}\mathcal{A}_{sing}$. The equality $-\delta^2_\varphi(\vec{K})\,\vec{\nabla}\varphi = \delta^1_\varphi(\vec{K}\times\vec{\nabla}\varphi)$ and Formula (3.23) lead to:

$$\begin{aligned} L\vec{\nabla}\varphi \in \delta^1_\varphi &\left(\vec{K}\times\vec{\nabla}\varphi + \sum_{\substack{j=0 \\ \varpi(u_j)\neq\varpi(\varphi)-|\varpi|}}^{\mu-1} \mathcal{C}_j\,u_j\,\vec{e}_\varpi \right) \\ &+ \bigoplus_{\substack{j=1 \\ \varpi(u_j)\neq\varpi(\varphi)-|\varpi|}}^{\mu-1} \operatorname{Cas}(\mathcal{A},\varphi)\vec{\nabla}u_j \oplus \bigoplus_{\substack{j=0 \\ \varpi(u_j)=\varpi(\varphi)-|\varpi|}}^{\mu-1} \operatorname{Cas}(\mathcal{A},\varphi)u_j\vec{\nabla}\varphi, \end{aligned} \tag{4.34}$$

where $\mathcal{C}_j \in \operatorname{Cas}(\mathcal{A},\varphi)$.

The equalities (4.33) and (4.34) give:

$$\begin{aligned} &\delta^1_\varphi\left(\vec{F} - \varphi\,\vec{H} - \sum_{\substack{j=1 \\ \varpi(u_j)=\varpi(\varphi)-|\varpi|}}^{\mu-1} c_j u_j\,\vec{e}_\varpi - \vec{K}\times\vec{\nabla}\varphi - \sum_{\substack{j=0 \\ \varpi(u_j)\neq\varpi(\varphi)-|\varpi|}}^{\mu-1} \mathcal{C}_j\,u_j\,\vec{e}_\varpi \right) \\ &\in \bigoplus_{\substack{j=1 \\ \varpi(u_j)\neq\varpi(\varphi)-|\varpi|}}^{\mu-1} \operatorname{Cas}(\mathcal{A},\varphi)\vec{\nabla}u_j \oplus \bigoplus_{\substack{j=0 \\ \varpi(u_j)=\varpi(\varphi)-|\varpi|}}^{\mu-1} \operatorname{Cas}(\mathcal{A},\varphi)u_j\vec{\nabla}\varphi. \end{aligned}$$

Using Proposition 3.19 once more, we obtain

$$\vec{F} - \varphi\,\vec{H} - \sum_{\substack{j=1 \\ \varpi(u_j)=\varpi(\varphi)-|\varpi|}}^{\mu-1} c_j u_j\,\vec{e}_\varpi - \vec{K} \times \vec{\nabla}\varphi - \sum_{\substack{j=0 \\ \varpi(u_j)\neq\varpi(\varphi)-|\varpi|}}^{\mu-1} \mathcal{C}_j\,u_j\,\vec{e}_\varpi \in Z^1(\mathcal{A},\varphi).$$

It suffices now to use Proposition 3.14 (the writing of $H^1(\mathcal{A},\varphi)$) to conclude that

$$\vec{F} \in \sum_{j=0}^{\mu-1} \mathbf{F}\,u_j\,\vec{e}_\varpi + \{\vec{\nabla}\varphi \times \vec{J} + \varphi\vec{L} \mid \vec{J},\vec{L} \in \mathcal{A}^3\}.$$

Finally, using Euler's Formula (3.5) and the definition of the u_j, it is easy the see that this sum is a direct one in $\mathcal{A}^3$. Hence the result for $H_2(\mathcal{A}_\varphi)$. $\square$

Remark 4.29 (Modular derivation). The Poisson variety $\mathcal{A}_\varphi$ that we have studied in this section is not a polynomial algebra $\mathbf{F}[x_1,\ldots,x_n]$, so that we can not apply the reasoning we have done in Section 2.2.3, we can not talk about modular derivation of the Poisson surface $(\mathcal{F}_\varphi,\{\cdot\,,\cdot\}_\varphi)$, because we have no volume form on $\mathcal{F}_\varphi$. We can only observ the results that we have obtained for the Poisson cohomology and homology spaces and see that there are no duality between these spaces.

5

Mathieu-Poisson homology

In [42], O. Mathieu has introduced, for any Poisson manifold, a homology with parameter. In this Chapter, we apply this definition to the case of an affine Poisson variety, see some interesting properties of this Mathieu-Poisson homology and determine it for the Poisson varieties we have studied in the previous chapters: $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$, $(\mathbf{F}^2, \{\cdot\,,\cdot\}^\psi)$ and $(\mathcal{F}_\varphi, \{\cdot\,,\cdot\}_{\mathcal{A}_\varphi})$. We will also compare the Mathieu-Poisson homology with the "classical" Poisson homology in these cases.

5.1 A homology with parameter

Let $(M, \pi = \{\cdot\,,\cdot\})$ be an affine Poisson variety and let $\mathcal{A}$ be its algebra of regular functions. In order to simplify the notations, we denote also by π the Poisson bracket $\{\cdot\,,\cdot\}$. We recall from Paragraph 2.2.2, that the "classical" Poisson homology complex of such a variety is defined as follows. The space of k-chains is the space of all Kähler differential k-forms of $\mathcal{A} = \mathcal{F}(M)$ and it is denoted by $\Omega^k(\mathcal{A})$, while the boundary operator, ∂_k^π, or simply $\partial_k : \Omega^k(\mathcal{A}) \to \Omega^{k-1}(\mathcal{A})$, called the Brylinsky or Koszul differential, is given by:

$$\begin{aligned}\partial_k(F_0\,\mathsf{d}F_1 \wedge \cdots \wedge \mathsf{d}F_k) &= \sum_{i=1}^{k}(-1)^{i+1}\,\{F_0, F_i\}\;\mathsf{d}F_1 \wedge \cdots \wedge \widehat{\mathsf{d}F_i} \wedge \ldots \wedge \mathsf{d}F_k \\ &+ \sum_{1\le i<j\le k}(-1)^{i+j}F_0\,\mathsf{d}\,\{F_i, F_j\} \wedge \mathsf{d}F_1 \wedge \cdots \wedge \widehat{\mathsf{d}F_i} \wedge \cdots \wedge \widehat{\mathsf{d}F_j} \wedge \cdots \wedge \mathsf{d}F_k,\end{aligned} \tag{5.1}$$

where the symbol $\widehat{\mathsf{d}F_i}$ means that we omit the term $\mathsf{d}F_i$. We have seen in Proposition 2.24, that an other way to write this operator is the following:

$$\partial_k = [\imath_\pi, \mathsf{d}] = \imath_\pi \circ \mathsf{d} - \mathsf{d} \circ \imath_\pi, \tag{5.2}$$

where $[\cdot\,,\cdot]$ is the graded commutator of graded linear maps.

In his article "Homologies associated with Poisson structures" [42], O. Mathieu defines a one parameter family of operators $(\partial_k^\tau : \Omega^k(\mathcal{A}) \to \Omega^{k-1}(\mathcal{A}))_{\tau\in\mathbf{F}}$ with:

$$\boxed{\partial_k^\tau = (\tau + k)\;\imath_\pi \circ \mathsf{d} - (\tau + k + 1)\;\mathsf{d} \circ \imath_\pi}\,. \tag{5.3}$$

Proposition 5.1. *For each $\tau \in \mathbf{F}$ and for all $k \in \mathbf{N}$, we have*

$$\partial_k^\tau \circ \partial_{k+1}^\tau = 0,$$

so that $\partial^\tau : \Omega^\bullet(\mathcal{A}) \to \Omega^{\bullet-1}(\mathcal{A})$ is a boundary operator.

Proof. This proof is the one given by O. Mathieu in [42]. Let us denote by Θ the operator $\Theta := -\partial \circ \mathsf{d} \circ \imath_\pi$. (In the following, in order to simplify the notations, we will sometimes omit the composition low "$\circ$".) By computing $\partial_k^\tau \circ \partial_{k+1}^\tau$ appear the operators $\mathsf{d}\imath_\pi\mathsf{d}\imath_\pi$, $\imath_\pi\mathsf{d}\imath_\pi\mathsf{d}$ and $\mathsf{d}\imath_\pi\imath_\pi\mathsf{d}$. We now want to express each of them, in terms of Θ. First, it is clear that

$$\mathsf{d}\imath_\pi\mathsf{d}\imath_\pi = -\partial\mathsf{d}\imath_\pi = \Theta,$$

then, using the properties of Proposition 2.24, we obtain:

$$\imath_\pi\mathsf{d}\imath_\pi\mathsf{d} = \imath_\pi\mathsf{d}\partial = -\partial\imath_\pi\mathsf{d} = -\partial\partial - \partial\mathsf{d}\imath_\pi = \Theta.$$

Finally, with the help of the two last equalities, we have

$$\mathsf{d}\imath_\pi\imath_\pi\mathsf{d} = -\partial\imath_\pi\mathsf{d} + \imath_\pi\mathsf{d}\imath_\pi\mathsf{d} = 2\Theta.$$

Now, it is easy to obtain

$$\begin{aligned}\partial_k^\tau \circ \partial_{k+1}^\tau &= (\tau+k)(\tau+k+1)\,\imath_\pi\mathsf{d}\imath_\pi\mathsf{d} - (\tau+k+1)^2\,\mathsf{d}\imath_\pi\imath_\pi\mathsf{d} \\ &\quad + (\tau+k+1)(\tau+k+2)\,\mathsf{d}\imath_\pi\mathsf{d}\imath_\pi \\ &= \big((\tau+k)(\tau+k+1) - 2(\tau+k+1)^2 + (\tau+k+1)(\tau+k+2)\big)\,\Theta \\ &= 0\end{aligned}$$

and ∂^τ is a boundary operator. □

We will call the operator ∂^τ the *Mathieu-Poisson boundary operator* and the corresponding homology is called the *Mathieu-Poisson homology*, or the *MP-homology*.

Remark 5.2. Let us consider, for all $\tau \in \mathbf{F}^*$ and $k \in \mathbf{N}$, the operator $\widetilde{\partial_k^\tau} : \Omega^k(\mathcal{A}) \to \Omega^{k-1}(\mathcal{A})$, given by

$$\widetilde{\partial_k^\tau} = \frac{\partial_k^\tau}{\tau} = \left(1 + \frac{k}{\tau}\right)\imath_\pi\mathsf{d} - \left(1 + \frac{k+1}{\tau}\right)\mathsf{d}\imath_\pi,$$

which is also a boundary operator. For any value of $\tau \in \mathbf{F}^*$, $\widetilde{\partial_k^\tau}$ and ∂_k^τ have the same cycles and the same boundaries, hence they define the same homology. When τ tends to infinity, $\widetilde{\partial_k^\tau}$ tends to ∂_k, so that, after normalization, ∂_k can be seen as the limit of the operators ∂_k^τ, for $\tau \to \infty$.

5.2 Some properties of MP-Homology

In this paragraph, we will give some properties of the MP-homology. In fact, we will point out some similarities and some differences between the classical Poisson homology and the MP-homology.

First, let us point out a difference between these notions.

Proposition 5.3. *Let $(M, \pi = \{\cdot\,,\cdot\})$ be a Poisson variety. Contrary to the case of classical Poisson boundary operator, the MP-boundary operator does not commute with the multiplication by a Casimir of $(M, \{\cdot\,,\cdot\})$, instead, for φ a Casimir and $\omega \in \Omega^\bullet(M)$,*

$$\partial^\tau(\varphi\,\omega) = \varphi\,\partial^\tau\omega - \mathsf{d}\varphi \wedge \imath_\pi\omega.$$

Proof. Let $(M, \pi = \{\cdot\,,\cdot\})$ be an affine Poisson variety and let $\varphi \in \mathcal{F}(M)$ be a Casimir for the Poisson bracket $\{\cdot\,,\cdot\}$, also denoted by π. Then, by definition of the inner product (See Paragraph 2.2.2), we have $\imath_\varphi\pi = 0$ and, according to Formula (2.23),

$$\imath_\pi(\mathsf{d}\varphi \wedge \omega) = \mathsf{d}\varphi \wedge \imath_\pi\omega.$$

So that, for all $\omega \in \Omega^k(M)$ $(k \in \mathbf{N})$, the MP-boundary operator gives:

$$\begin{aligned}\partial_k^\tau(\varphi\,\omega) &= (\tau + k)\;\imath_\pi \circ \mathsf{d}\,(\varphi\omega) - (\tau + k + 1)\;\mathsf{d} \circ \imath_\pi\,(\varphi\omega)\\ &= (\tau + k)\;\imath_\pi\,(\mathsf{d}\varphi \wedge \omega) + (\tau + k)\;\varphi\,\imath_\pi \circ \mathsf{d}\omega\\ &\quad -(\tau + k + 1)\;\mathsf{d}\varphi \wedge \imath_\pi\omega - (\tau + k + 1)\;\varphi\,\mathsf{d} \circ \imath_\pi\omega\\ &= \varphi\,\partial_k^\tau\omega - \mathsf{d}\varphi \wedge \imath_\pi\omega,\end{aligned}$$

hence the result. □

It follows that the spaces of k-cycles and k-boundaries are not modules over the Casimirs. We will see in Section 5.3 that, in general, the MP-homology spaces are also not modules over the Casimirs.

As we have seen in Remark 2.21, the classical Poisson homology is a contravariant functor. We have the same result for the MP-homology. We have indeed seen in this remark, that, for any morphism of affine Poisson varieties $\varphi : (M_1, \pi_1) \to (M_2, \pi_2)$, the dual map $\varphi^* : \mathcal{F}(M_2) \to \mathcal{F}(M_1)$ and its extension $\Omega^\bullet(\varphi^*) : \Omega^\bullet(\mathcal{F}(M_2)) \to \Omega^\bullet(\mathcal{F}(M_1))$ commutes with d and $\Omega^\bullet(\varphi^*) \circ \imath_{\pi_2} = \imath_{\pi_1} \circ \Omega^\bullet(\varphi^*)$. This implies that $\Omega^\bullet(\varphi^*) \circ \partial^\tau_{\pi_2} = \partial^\tau_{\pi_1} \circ \Omega^\bullet(\varphi^*)$ and $H^\tau_\bullet(\varphi^*) : H^\tau_\bullet(M_2, \pi_2) \to H^\tau_\bullet(M_1, \pi_1)$ is a homomorphism of graded vector spaces. Thus we have obtained the following:

Proposition 5.4. *The MP-homology is a contravariant functor between the category of affine Poisson varieties and the category of graded vector spaces.*

Finally, we point out that in his article [42], O. Mathieu shows that, for generic values of the parameter τ, the MP-homology of a symplectic manifold is isomorphic to its classical Poisson homology.

5.3 MP-homology of the Poisson variety $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$

Now, we want to consider the MP-homology of the Poisson variety $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$, where we recall that, for $\varphi \in \mathcal{A} := \mathbf{F}[x, y, z]$,

$$\{\cdot\,,\cdot\}_\varphi = \frac{\partial\varphi}{\partial x}\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + \frac{\partial\varphi}{\partial y}\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + \frac{\partial\varphi}{\partial z}\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}. \tag{5.4}$$

We have indeed seen in Paragraph 2.1.3 that, for any $\varphi \in \mathcal{A}$, the bracket $\{\cdot\,,\cdot\}_\varphi$ is a Poisson structure on $\mathbf{F}^3$. In Section 3.2, we have computed the classical Poisson homology of $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$, for $\varphi \in \mathcal{A}$, a weight homogeneous polynomial with an isolated singularity (at the origin). Our purpose, in this section, is to determine the MP-homology of these Poisson varieties, for generic values of the parameter τ. We will be inspired by Paragraphes 3.1.1 and 3.1.2, identifying the spaces

$$\Omega^0(\mathcal{A}) \simeq \Omega^3(\mathcal{A}) \simeq \mathcal{A} \qquad \Omega^1(\mathcal{A}) \simeq \Omega^2(\mathcal{A}) \simeq \mathcal{A}^3$$

and using the notations of vector calculus in $\mathbf{R}^3$, adapted to $\mathcal{A}^3$: elements of $\mathcal{A}^3 \simeq \Omega^1(\mathcal{A}) \simeq \Omega^2(\mathcal{A})$ will be denoted with an arrow, like $\vec{F} \in \Omega^1(\mathcal{A})$, or $\vec{F} \in \Omega^2(\mathcal{A})$. We will use the inner and the cross products $\cdot$ and $\times$ and the gradient, the curl and the divergence operators $\vec{\nabla}$, $\vec{\nabla}\times$ and Div. Then, we write the MP-boundary operator in terms of $\mathcal{A}$ and $\mathcal{A}^3$. For example, for $\vec{F} = (F_1, F_2, F_3) \in \mathcal{A}^3 \simeq \Omega^1(\mathcal{A})$, i.e., $\vec{F} = F_1\mathsf{d}x + F_2\mathsf{d}y + F_3\mathsf{d}z$, we have $\imath_\pi(\vec{F}) = 0$, so that, $\partial_1^\tau(\vec{F}) = (\tau + 1)\,\imath_\pi\mathsf{d}\,(F_1\mathsf{d}x + F_2\mathsf{d}y + F_3\mathsf{d}z)$, and

$$\begin{aligned}
&\imath_\pi\mathsf{d}\,(F_1\mathsf{d}x + F_2\mathsf{d}y + F_3\mathsf{d}z)\\
&= \imath_\pi\Bigg(\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathsf{d}y \wedge \mathsf{d}z + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathsf{d}z \wedge \mathsf{d}x\\
&\qquad + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathsf{d}x \wedge \mathsf{d}y\Bigg)\\
&= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\{y, z\}_\varphi + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\{z, x\}_\varphi\\
&\qquad + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\{x, y\}_\varphi\\
&= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\frac{\partial\varphi}{\partial x} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\frac{\partial\varphi}{\partial y} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\frac{\partial\varphi}{\partial z}\\
&= \vec{\nabla}\varphi \cdot \left(\vec{\nabla} \times \vec{F}\right).
\end{aligned}$$

By doing an analogous reasoning for ∂_2^τ and ∂_3^τ, we can write:

$$\begin{aligned}
&\partial_0^\tau(F) = 0, \qquad \text{for } F \in \mathcal{A} \simeq \Omega^0(\mathcal{A}),\\
&\partial_1^\tau(\vec{F}\,) = (\tau+1)\,\vec{\nabla}\varphi \cdot (\vec{\nabla} \times \vec{F}), \qquad \text{for } \vec{F} \in \mathcal{A}^3 \simeq \Omega^1(\mathcal{A}),\\
&\partial_2^\tau(\vec{F}\,) = (\tau+2)\,\mathrm{Div}(\vec{F})\vec{\nabla}\varphi - (\tau+3)\,\vec{\nabla}(\vec{F}\cdot\vec{\nabla}\varphi),\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \vec{F} \in \mathcal{A}^3 \simeq \Omega^2(\mathcal{A}),\\
&\partial_3^\tau(F) = -(\tau+4)\,\vec{\nabla}F \times \vec{\nabla}\varphi, \qquad \text{for } F \in \mathcal{A} \simeq \Omega^3(\mathcal{A}),
\end{aligned} \tag{5.5}$$

while, for the classical Poisson homology, we recall that, according to (3.33) and (3.14), the boundary operator ∂_k, associated to the Poisson variety $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$ is of the following form:

$$\begin{aligned}
&\partial_0(F) = 0, \qquad \text{for } F \in \mathcal{A} \simeq \Omega^0(\mathcal{A}),\\
&\partial_1(\vec{F}\,) = \vec{\nabla}\varphi \cdot (\vec{\nabla} \times \vec{F}), \qquad \text{for } \vec{F} \in \mathcal{A}^3 \simeq \Omega^1(\mathcal{A}),\\
&\partial_2(\vec{F}\,) = \mathrm{Div}(\vec{F})\vec{\nabla}\varphi - \vec{\nabla}(\vec{F}\cdot\vec{\nabla}\varphi), \qquad \text{for } \vec{F} \in \mathcal{A}^3 \simeq \Omega^2(\mathcal{A}),\\
&\partial_3(F) = -\vec{\nabla}F \times \vec{\nabla}\varphi, \qquad \text{for } F \in \mathcal{A} \simeq \Omega^3(\mathcal{A}).
\end{aligned}$$

We denote by $B_i^\tau(\mathcal{A},\varphi) = \mathrm{Im}(\partial_{i+1}^\tau)$, the space of all i-boundaries and by $Z_i^\tau(\mathcal{A},\varphi) = \ker \partial_i^\tau$, the space of all i-cycles. The MP-homology spaces of the Poisson variety $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$ take the following forms

$$\begin{aligned}
&H_0^\tau(\mathcal{A},\varphi) \simeq \frac{\mathcal{A}}{\left\{(\tau+1)\,\vec{\nabla}\varphi\cdot(\vec{\nabla}\times\vec{F}) \mid \vec{F}\in\mathcal{A}^3\right\}},\\
&H_1^\tau(\mathcal{A},\varphi) \simeq \frac{\left\{\vec{F}\in\mathcal{A}^3 \mid (\tau+1)\,\vec{\nabla}\varphi\cdot(\vec{\nabla}\times\vec{F}) = 0\right\}}{\left\{(\tau+2)\ \mathrm{Div}(\vec{F})\vec{\nabla}\varphi - (\tau+3)\,\vec{\nabla}(\vec{F}\cdot\vec{\nabla}\varphi) \mid \vec{F}\in\mathcal{A}^3\right\}},\\
&H_2^\tau(\mathcal{A},\varphi) \simeq \frac{\left\{\vec{F}\in\mathcal{A}^3 \mid (\tau+2)\ \mathrm{Div}(\vec{F})\vec{\nabla}\varphi - (\tau+3)\,\vec{\nabla}(\vec{F}\cdot\vec{\nabla}\varphi) = \vec{0}\right\}}{\left\{(\tau+4)\,\vec{\nabla}F\times\vec{\nabla}\varphi \mid F\in\mathcal{A}\right\}},\\
&H_3^\tau(\mathcal{A},\varphi) \simeq \left\{F\in\mathcal{A} \mid (\tau+4)\,\vec{\nabla}F\times\vec{\nabla}\varphi = \vec{0}\right\}.
\end{aligned}$$

For generic values of τ (i.e., $\tau \neq -1, -4$), these formulas simplify to:

$$H_0^\tau(\mathcal{A},\varphi) \simeq \frac{\mathcal{A}}{\left\{\vec{\nabla}\varphi\cdot(\vec{\nabla}\times\vec{F}) \mid \vec{F}\in\mathcal{A}^3\right\}} \simeq H_0(\mathcal{A},\varphi)$$

$$H_1^\tau(\mathcal{A},\varphi) \simeq \frac{\left\{\vec{F}\in\mathcal{A}^3 \mid \vec{\nabla}\varphi\cdot(\vec{\nabla}\times\vec{F})=0\right\}}{\left\{(\tau+2)\,\mathrm{Div}(\vec{F})\vec{\nabla}\varphi-(\tau+3)\,\vec{\nabla}(\vec{F}\cdot\vec{\nabla}\varphi) \mid \vec{F}\in\mathcal{A}^3\right\}},$$

$$H_2^\tau(\mathcal{A},\varphi) \simeq \frac{\left\{\vec{F}\in\mathcal{A}^3 \mid (\tau+2)\,\mathrm{Div}(\vec{F})\vec{\nabla}\varphi-(\tau+3)\,\vec{\nabla}(\vec{F}\cdot\vec{\nabla}\varphi)=\vec{0}\right\}}{\left\{\vec{\nabla}F\times\vec{\nabla}\varphi \mid F\in\mathcal{A}\right\}},$$

$$H_3^\tau(\mathcal{A},\varphi) \simeq \left\{F\in\mathcal{A} \mid \vec{\nabla}F\times\vec{\nabla}\varphi=\vec{0}\right\} \simeq H_3(\mathcal{A},\varphi),$$

so that, for $\tau \neq -1,-4$, we have already obtained (See Paragraphs 3.2.2 and 3.2.4),

$$H_3^\tau(\mathcal{A},\varphi) \simeq H_3(\mathcal{A},\varphi) \simeq \mathrm{Cas}(\mathcal{A},\varphi) \simeq \bigoplus_{i\in\mathbf{N}}\mathbf{F}\varphi^i,$$

$$H_0^\tau(\mathcal{A},\varphi) \simeq H_0(\mathcal{A},\varphi) \simeq \mathrm{Cas}(\mathcal{A},\varphi)\otimes_{\mathbf{F}}\mathcal{A}_{sing},$$

where we recall that

$$\mathcal{A}_{sing} = \frac{\mathbf{F}[x,y,z]}{\langle\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial y},\frac{\partial\varphi}{\partial z}\rangle}$$

is the singular algebra associated to φ (See Paragraph 2.3.2). In Paragraph 2.3.2, we have seen that saying that φ is weight homogeneous, with an isolated singularity, is equivalent to saying that $\mathcal{A}_{sing}$, viewed as a $\mathbf{F}$-vector space is of finite dimension, denoted by μ (the Milnor number of the singularity).

We will use the results and the methods of Chapter 3, where we have studied the Poisson cohomology of the Poisson variety $(\mathbf{F}^3,\{\cdot\,,\cdot\}_\varphi)$, to determine the MP-homology spaces $H_1^\tau(\mathcal{A},\varphi)$ and $H_2^\tau(\mathcal{A},\varphi)$, for generic values of τ, i.e., for $\tau\notin\{-1,-2,-3,-4\}$. We first need the following:

Lemma 5.5. *For $r\in\mathbf{N}$ and $j=0,\ldots,\mu-1$,*

$$\begin{aligned}\partial_2^\tau(\varphi^r u_j\vec{e}_\varpi) = (\tau+2)(\varpi(u_j)-(\varpi(\varphi)D_\tau^{r+1}-|\varpi|))\;\varphi^r u_j\vec{\nabla}\varphi\\ -(\tau+3)\varpi(\varphi)\;\varphi^{r+1}\vec{\nabla}u_j.\end{aligned} \tag{5.6}$$

In particular, denoting by D_τ^r, for all $r\in\mathbf{N}$, the element $D_\tau^r := \frac{r}{\tau+2}+1\in\mathbf{F}$,

$$\textit{if } r\geq 1 \textit{ and } \varpi(u_j)=\varpi(\varphi)D_\tau^r-|\varpi|, \textit{ then } \varphi^r\vec{\nabla}u_j\in B_1^\tau(\mathcal{A},\varphi).$$

Moreover, for $j=0$ and $r\in\mathbf{N}$, we obtain:

$$\partial_2^\tau(\varphi^r \vec{e}_\varpi) = -(\tau+2)(\varpi(\varphi) D_\tau^{r+1} - |\varpi|)\, \varphi^r \vec{\nabla}\varphi, \tag{5.7}$$

while, for $r = 0$ and $j \geq 0$,

$$\begin{aligned}\partial_2^\tau(u_j \vec{e}_\varpi) = ((\tau+2)(\varpi(u_j) + |\varpi|) - (\tau+3)\varpi(\varphi))\, u_j \vec{\nabla}\varphi \\ -(\tau+3)\varpi(\varphi)\, \varphi \vec{\nabla} u_j.\end{aligned} \tag{5.8}$$

Proof. Let $r \in \mathbf{N}$ and $j = 0, \ldots, \mu - 1$,

$$\begin{aligned}
&\partial_2^\tau(\varphi^r u_j \vec{e}_\varpi) = -(\tau+3)\varpi(\varphi)\, \vec{\nabla}(\varphi^{r+1} u_j) + (\tau+2)\operatorname{Div}(\varphi^r u_j \vec{e}_\varpi)\vec{\nabla}\varphi \\
&= -(\tau+3)\varpi(\varphi)\, \vec{\nabla}(\varphi^{r+1} u_j) + (\tau+2)(\varpi(\varphi) r + \varpi(u_j) + |\varpi|)\, \varphi^r u_j \vec{\nabla}\varphi \\
&= (-(\tau+3)\varpi(\varphi)(r+1) + (\tau+2)(\varpi(\varphi) r + \varpi(u_j) + |\varpi|))\, \varphi^r u_j \vec{\nabla}\varphi \\
&\quad -(\tau+3)\varpi(\varphi)\, \varphi^{r+1}\vec{\nabla} u_j \\
&= (-\varpi(\varphi)(r+\tau+3) + (\tau+2)(\varpi(u_j) + |\varpi|))\, \varphi^r u_j \vec{\nabla}\varphi \\
&\quad -(\tau+3)\varpi(\varphi)\, \varphi^{r+1}\vec{\nabla} u_j \\
&= (\tau+2)(\varpi(u_j) - (\varpi(\varphi) D_\tau^{r+1} - |\varpi|))\, \varphi^r u_j \vec{\nabla}\varphi \\
&\quad -(\tau+3)\varpi(\varphi)\, \varphi^{r+1}\vec{\nabla} u_j,
\end{aligned}$$

hence the result. □

We now have the:

Proposition 5.6. *Let $\varphi \in \mathcal{A} = \mathbf{F}[x,y,z]$ be a weight homogeneous polynomial with an isolated singularity. As in Paragraph 3.2.4, we denote by $u_0 = 1, u_1, \ldots, u_{\mu-1} \in \mathcal{A}$ a family of weight homogeneous polynomials of $\mathcal{A}$, such that their projections in $\mathcal{A}_{sing}$ give a $\mathbf{F}$-basis of this $\mathbf{F}$-vector space.*

For $\tau \in \mathbf{F} \setminus \{-1,-2,-3,-4\}$, the first MP-homology space $H_1^\tau(\mathcal{A},\varphi)$ of the Poisson variety $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$ is given by:

$$\begin{aligned}
H_1^\tau(\mathcal{A},\varphi) \simeq \bigoplus_{r\in\mathbf{N}} \bigoplus_{\substack{j=1 \\ \varpi(u_j)\neq\varpi(\varphi)D_\tau^r - |\varpi|}}^{\mu-1} \mathbf{F}\varphi^r \vec{\nabla} u_j \oplus \bigoplus_{r\in\mathbf{N}} \bigoplus_{\substack{j=0 \\ \varpi(u_j)=\varpi(\varphi)D_\tau^{r+1} - |\varpi|}}^{\mu-1} \mathbf{F}\varphi^r u_j \vec{\nabla}\varphi \\
\oplus \bigoplus_{\substack{j=1 \\ \varpi(u_j)=\varpi(\varphi) - |\varpi|}}^{\mu-1} \mathbf{F}\vec{\nabla} u_j,
\end{aligned}$$

where, for all $r \in \mathbf{N}$, D_τ^r denotes, $D_\tau^r := \dfrac{r}{(\tau+2)} + 1 \in \mathbf{F}$.

Remark 5.7. Notice that, if τ tends to ∞, then the condition $\varpi(u_j) = \varpi(\varphi)D_\tau^{r+1} - |\varpi|$ tends to $\varpi(u_j) = \varpi(\varphi) - |\varpi|$ and is independent of the r, so that, according to Proposition 3.19 and Proposition 3.23, the Proposition 5.6 holds for $\tau \to \infty$, if we see the classical operator as the limit operator ∂_k^∞ (See Remark 5.2).

Proof. This proof is an adapted copy of the one of the determination of the Poisson cohomology space $H^2(\mathcal{A}, \varphi)$ in Proposition 3.19. Let $\tau \in \mathbf{F} \setminus \{-1, -2, -3, -4\}$. First, we will show that:

$$\begin{aligned} Z_1^\tau(\mathcal{A}, \varphi) \subseteq B_1^\tau(\mathcal{A}, \varphi) &+ \sum_{r \in \mathbf{N}} \sum_{\substack{j=1 \\ \varpi(u_j) \neq \varpi(\varphi)D_\tau^r - |\varpi|}}^{\mu-1} \mathbf{F}\varphi^r \vec{\nabla} u_j \\ &+ \sum_{r \in \mathbf{N}} \sum_{\substack{j=0 \\ \varpi(u_j) = \varpi(\varphi)D_\tau^{r+1} - |\varpi|}}^{\mu-1} \mathbf{F}\varphi^r u_j \vec{\nabla}\varphi \quad + \sum_{\substack{j=1 \\ \varpi(u_j) = \varpi(\varphi) - |\varpi|}}^{\mu-1} \mathbf{F}\vec{\nabla} u_j. \end{aligned} \tag{5.9}$$

Let $\vec{F} \in \mathcal{A}^3$, satisfying the cocycle condition: $\vec{\nabla}\varphi \cdot (\vec{\nabla} \times \vec{F}) = 0$. Then, according to Corollary 3.9, there exist $G, H \in \mathcal{A}$ such that

$$\vec{F} = \vec{\nabla} G + H \vec{\nabla}\varphi. \tag{5.10}$$

We point out that, because $\tau \in \mathbf{F} \setminus \{-2, -3\}$, for all $\vec{L} \in \mathcal{A}^3$, the two elements $\vec{\nabla}(\partial_1^\tau(\vec{L}\,))$ and $\partial_1^\tau(\vec{L}\,)\,\vec{\nabla}\varphi$ are in $B_1^\tau(\mathcal{A}, \varphi)$, as we have:

$$\vec{\nabla}(\partial_1^\tau(\vec{L}\,)) = (\tau+1)\vec{\nabla}\left(\left(\vec{\nabla} \times \vec{L}\,\right) \cdot \vec{\nabla}\varphi\right) = -\frac{\tau+1}{\tau+3}\,\partial_2^\tau\left(\vec{\nabla} \times \vec{L}\,\right),$$

and:

$$\begin{aligned} \partial_1^\tau(\vec{L}\,)\,\vec{\nabla}\varphi = (\tau+1)\left(\vec{\nabla}\varphi \cdot \left(\vec{\nabla} \times \vec{L}\,\right)\right)\,\vec{\nabla}\varphi &= (\tau+1)\left(\mathrm{Div}\left(\vec{L} \times \vec{\nabla}\varphi\right)\right)\,\vec{\nabla}\varphi \\ &= \frac{\tau+1}{\tau+2}\,\partial_2^\tau\left(\vec{L} \times \vec{\nabla}\varphi\right). \end{aligned}$$

Moreover, the writing of $H_0^\tau(\mathcal{A}, \varphi)$ permits us to write (with $\vec{L}, \vec{K} \in \mathcal{A}^3$):

$$G \in \partial_1^\tau(\vec{L}\,) + \sum_{j=1}^{\mu-1} \mathrm{Cas}(\mathcal{A}, \varphi) u_j, \quad H \in \partial_1^\tau(\vec{K}\,) + \sum_{j=1}^{\mu-1} \mathrm{Cas}(\mathcal{A}, \varphi) u_j,$$

thus, we obtain:

$$\begin{aligned} \vec{F} &\in B_1^\tau(\mathcal{A}, \varphi) + \sum_{j=1}^{\mu-1} \mathrm{Cas}(\mathcal{A}, \varphi)\vec{\nabla} u_j + \sum_{j=0}^{\mu-1} \mathrm{Cas}(\mathcal{A}, \varphi) u_j \vec{\nabla}\varphi \\ &\in B_1^\tau(\mathcal{A}, \varphi) + \sum_{r \in \mathbf{N}} \sum_{j=1}^{\mu-1} \mathbf{F}\varphi^r \vec{\nabla} u_j + \sum_{r \in \mathbf{N}} \sum_{j=0}^{\mu-1} \mathbf{F}\varphi^r u_j \vec{\nabla}\varphi, \end{aligned} \tag{5.11}$$

so that:

$$Z_1^\tau(\mathcal{A},\varphi) \subseteq B_1^\tau(\mathcal{A},\varphi) + \sum_{r\in\mathbf{N}}\sum_{j=1}^{\mu-1}\mathbf{F}\varphi^r\vec{\nabla}u_j + \sum_{r\in\mathbf{N}}\sum_{j=0}^{\mu-1}\mathbf{F}\varphi^r u_j\vec{\nabla}\varphi$$

Now, we use Lemma 5.5, to obtain the desired inclusion (5.9). According to Equation (5.6), if $\varpi(u_j) \neq \varpi(\varphi)D_\tau^{r+1} - |\varpi|$, then, $\mathbf{F}\varphi^r u_j\vec{\nabla}\varphi \in B_1^\tau(\mathcal{A},\varphi) + \mathbf{F}\varphi^{r+1}\vec{\nabla}u_j$, so that we have:

$$Z_1^\tau(\mathcal{A},\varphi) \subseteq B_1^\tau(\mathcal{A},\varphi) + \sum_{r\in\mathbf{N}}\sum_{j=1}^{\mu-1}\mathbf{F}\varphi^r\vec{\nabla}u_j + \sum_{r\in\mathbf{N}}\sum_{\substack{j=0\\ \varpi(u_j)=\varpi(\varphi)D_\tau^{r+1}-|\varpi|}}^{\mu-1}\mathbf{F}\varphi^r u_j\vec{\nabla}\varphi;$$

moreover, if $r > 0$ and $\varpi(u_j) = \varpi(\varphi)D_\tau^r - |\varpi|$, then $\varphi^r\vec{\nabla}u_j \in B^\tau(\mathcal{A},\varphi)$, so that we obtain:

$$\begin{aligned} Z_1^\tau(\mathcal{A},\varphi) \subseteq B_1^\tau(\mathcal{A},\varphi) &+ \sum_{r\in\mathbf{N}}\sum_{\substack{j=1\\ \varpi(u_j)\neq\varpi(\varphi)D_\tau^{r}-|\varpi|}}^{\mu-1}\mathbf{F}\varphi^r\vec{\nabla}u_j + \sum_{\substack{j=1\\ \varpi(u_j)=\varpi(\varphi)D_\tau^{0}-|\varpi|}}^{\mu-1}\mathbf{F}\vec{\nabla}u_j \\ &+ \sum_{r\in\mathbf{N}}\sum_{\substack{j=0\\ \varpi(u_j)=\varpi(\varphi)D_\tau^{r+1}-|\varpi|}}^{\mu-1}\mathbf{F}\varphi^r u_j\vec{\nabla}\varphi. \end{aligned}$$

Finally, by pointing out that $D_\tau^0 = 1$, we obtain the decomposition (5.9).

Now, it is clear that we have the other inclusion, and it remains to show that this sum is a direct one, modulo $B_1^\tau(\mathcal{A},\varphi)$.

For this purpose, let us consider some elements of $\mathbf{F}$: $(\lambda_j, \gamma_{r,l}, \delta_{i,k})$, for $1 \le j \le \mu - 1$ such that $\varpi(u_j) = \varpi(\varphi) - |\varpi|$, $(i,k) \in \mathbf{N}\times\{0,\ldots,\mu-1\}$, with $\varpi(u_k) = \varpi(\varphi)D_\tau^{i+1} - |\varpi|$, $(r,l) \in \mathbf{N}\times\{1,\ldots,\mu-1\}$, such that $\varpi(u_l) \neq \varpi(\varphi)D_\tau^r - |\varpi|$ and an element $\vec{H} \in \mathcal{A}^3$ satisfying the equation in $\Omega^1(\mathcal{A}) \simeq \mathcal{A}^3$:

$$\begin{aligned} &\sum_{r\geq 0}\sum_{\substack{l=1\\ \varpi(u_l)\neq\varpi(\varphi)D_\tau^{r}-|\varpi|}}^{\mu-1}\gamma_{r,l}\,\varphi^r\vec{\nabla}u_l + \sum_{i\geq 0}\sum_{\substack{k=0\\ \varpi(u_k)=\varpi(\varphi)D_\tau^{i+1}-|\varpi|}}^{\mu-1}\delta_{i,k}\,\varphi^i u_k\vec{\nabla}\varphi \\ &+ \sum_{\substack{j=1\\ \varpi(u_j)=\varpi(\varphi)-|\varpi|}}^{\mu-1}\lambda_j\,\vec{\nabla}u_j = \partial_2^\tau(\vec{H}) = -(\tau+3)\vec{\nabla}(\vec{H}\cdot\vec{\nabla}\varphi) + (\tau+2)\,\mathrm{Div}(\vec{H})\vec{\nabla}\varphi \end{aligned}$$

(we consider only finite sums, as, for i and r great, $\gamma_{r,l} = \delta_{i,k} = 0$). Taking the inner product of this equality with the Euler derivation $\vec{e}_\varpi$, the definition of the u_j, leads to $\gamma_{0,l} = 0$, for all l, and $\lambda_j = 0$, for all j. We so obtain:

$$\sum_{r\geq 1}\sum_{\substack{l=1\\ \varpi(u_l)\neq\varpi(\varphi)D_\tau^r-|\varpi|}}^{\mu-1}\gamma_{r,l}\,\varphi^r\vec{\nabla}u_l+\sum_{i\geq 0}\sum_{\substack{k=0\\ \varpi(u_k)=\varpi(\varphi)D_\tau^{i+1}-|\varpi|}}^{\mu-1}\delta_{i,k}\,\varphi^i u_k\vec{\nabla}\varphi$$
$$=\partial_2^\tau(\vec{H})=-(\tau+3)\vec{\nabla}(\vec{H}\cdot\vec{\nabla}\varphi)+(\tau+2)\,\mathrm{Div}(\vec{H})\vec{\nabla}\varphi,$$

that we write as follows:

$$\vec{\nabla}\left(\sum_{r\geq 1}\sum_{\substack{l=1\\ \varpi(u_l)\neq\varpi(\varphi)D_\tau^r-|\varpi|}}^{\mu-1}\gamma_{r,l}\,\varphi^r u_l+(\tau+3)\,\vec{H}\cdot\vec{\nabla}\varphi\right)=H\vec{\nabla}\varphi, \tag{5.12}$$

where

$$H:=\sum_{r\geq 1}\sum_{\substack{l=1\\ \varpi(u_l)\neq\varpi(\varphi)D_\tau^r-|\varpi|}}^{\mu-1}r\,\gamma_{r,l}\,\varphi^{r-1}u_l$$
$$-\sum_{i\geq 0}\sum_{\substack{k=0\\ \varpi(u_k)=\varpi(\varphi)D_\tau^{i+1}-|\varpi|}}^{\mu-1}\delta_{i,k}\,\varphi^i u_k+(\tau+2)\,\mathrm{Div}(\vec{H}).$$

Moreover, the Equation (5.12) leads to the fact that the element

$$\sum_{r\geq 1}\sum_{\substack{l=1\\ \varpi(u_l)\neq\varpi(\varphi)D_\tau^r-|\varpi|}}^{\mu-1}\gamma_{r,l}\,\varphi^r u_l+(\tau+3)\,\vec{H}\cdot\vec{\nabla}\varphi$$

is a Casimir of $(\mathbf{F}^3,\{\cdot\,,\cdot\}_\varphi)$, so that, according to Proposition 3.11, there exist some elements c_s, $s\geq 1$, satisfying:

$$\sum_{r\geq 1}\sum_{\substack{l=1\\ \varpi(u_l)\neq\varpi(\varphi)D_\tau^r-|\varpi|}}^{\mu-1}\gamma_{r,l}\,\varphi^r u_l+(\tau+3)\,\vec{H}\cdot\vec{\nabla}\varphi=\sum_{s\geq 1}c_s\,\varphi^s. \tag{5.13}$$

That leads to:

$$(\tau+3)\,\vec{H}\cdot\vec{\nabla}\varphi=\sum_{s\geq 1}c_s\varphi^s-\sum_{r\geq 1}\sum_{\substack{l=1\\ \varpi(u_l)\neq\varpi(\varphi)D_\tau^r-|\varpi|}}^{\mu-1}\gamma_{r,l}\varphi^r u_l$$
$$=\left(\sum_{s\geq 1}\frac{c_s}{\varpi(\varphi)}\varphi^{s-1}\vec{e}_\varpi-\sum_{r\geq 1}\sum_{\substack{l=1\\ \varpi(u_l)\neq\varpi(\varphi)D_\tau^r-|\varpi|}}^{\mu-1}\frac{\gamma_{r,l}}{\varpi(\varphi)}\varphi^{r-1}u_l\,\vec{e}_\varpi\right)\cdot\vec{\nabla}\varphi.$$

The exactness of the Koszul complex (See Proposition 3.5) gives the existence of an element $\vec{K}\in\mathcal{A}^3$ such that:

$$(\tau+3)\,\vec{H} = \sum_{s\geq 1} \frac{c_s}{\varpi(\varphi)}\varphi^{s-1}\vec{e}_\varpi - \sum_{r\geq 1} \sum_{\substack{l=1\\ \varpi(u_l)\neq\varpi(\varphi)D_\tau^r-|\varpi|}}^{\mu-1} \frac{\gamma_{r,l}}{\varpi(\varphi)}\varphi^{r-1}u_l\,\vec{e}_\varpi + \vec{K}\times\vec{\nabla}\varphi.$$

That permits us to compute:

$$\begin{aligned}(\tau+3)\ \mathrm{Div}(\vec{H}\ ) &= \sum_{s\geq 1} \frac{c_s}{\varpi(\varphi)}(\varpi(\varphi)(s-1)+|\varpi|)\varphi^{s-1}\\ &- \sum_{r\geq 1} \sum_{\substack{l=1\\ \varpi(u_l)\neq\varpi(\varphi)D_\tau^r-|\varpi|}}^{\mu-1} \frac{\gamma_{r,l}}{\varpi(\varphi)}(\varpi(\varphi)(r-1)+\varpi(u_l)+|\varpi|)\ \varphi^{r-1}u_l\\ &+ (\vec{\nabla}\times\vec{K})\cdot\vec{\nabla}\varphi.\end{aligned}$$

Then, Equations above (5.12) and (5.13) imply:

$$\begin{aligned}\sum_{s\geq 1} s\,c_s\ \varphi^{s-1} &= \sum_{r\geq 1} \sum_{\substack{l=1\\ \varpi(u_l)\neq\varpi(\varphi)D_\tau^r-|\varpi|}}^{\mu-1} r\,\gamma_{r,l}\ \varphi^{r-1}u_l\\ &- \sum_{i\geq 0} \sum_{\substack{k=0\\ \varpi(u_k)=\varpi(\varphi)D_\tau^{i+1}-|\varpi|}}^{\mu-1} \delta_{i,k}\ \varphi^i u_k\\ &+ \frac{(\tau+2)}{(\tau+3)}\sum_{s\geq 1} \frac{c_s}{\varpi(\varphi)}(\varpi(\varphi)(s-1)+|\varpi|)\varphi^{s-1}\\ &- \frac{(\tau+2)}{(\tau+3)}\sum_{r\geq 1} \sum_{\substack{l=1\\ \varpi(u_l)\neq\varpi(\varphi)D_\tau^r-|\varpi|}}^{\mu-1} \frac{\gamma_{r,l}}{\varpi(\varphi)}(\varpi(\varphi)(r-1)+\varpi(u_l)+|\varpi|)\ \varphi^{r-1}u_l\\ &+ \frac{(\tau+2)}{(\tau+3)}(\vec{\nabla}\times\vec{K})\cdot\vec{\nabla}\varphi,\end{aligned}$$

that is to say:

$$\begin{aligned}&\frac{(\tau+2)}{(\tau+3)(\tau+1)}\partial_1^\tau(\vec{K}) = \frac{(\tau+2)}{(\tau+3)}(\vec{\nabla}\times\vec{K})\cdot\vec{\nabla}\varphi\\ &= \sum_{s\geq 1} \frac{c_s}{\varpi(\varphi)}\frac{(\tau+2)}{(\tau+3)}(\varpi(\varphi)D_\tau^s-|\varpi|)\ \varphi^{s-1} + \sum_{i\geq 0} \sum_{\substack{k=0\\ \varpi(u_k)=\varpi(\varphi)D_\tau^{i+1}-|\varpi|}}^{\mu-1} \delta_{i,k}\ \varphi^i u_k\\ &+ \frac{(\tau+2)}{(\tau+3)}\sum_{r\geq 1} \sum_{\substack{l=1\\ \varpi(u_l)\neq\varpi(\varphi)D_\tau^r-|\varpi|}}^{\mu-1} \frac{\gamma_{r,l}}{\varpi(\varphi)}(\varpi(u_l)-\varpi(\varphi)D_\tau^r+|\varpi|)\ \varphi^{r-1}u_l\end{aligned}$$

$$= \sum_{s\geq 0} \frac{c_{s+1}}{\varpi(\varphi)} \frac{(\tau+2)}{(\tau+3)} (\varpi(\varphi) D_\tau^{s+1} - |\varpi|)\, \varphi^s + \sum_{i\geq 0} \sum_{\substack{k=0 \\ \varpi(u_k)=\varpi(\varphi)D_\tau^{i+1}-|\varpi|}}^{\mu-1} \delta_{i,k}\, \varphi^i u_k$$

$$+ \frac{(\tau+2)}{(\tau+3)} \sum_{r\geq 0} \sum_{\substack{l=1 \\ \varpi(u_l)\neq\varpi(\varphi)D_\tau^{r+1}-|\varpi|}}^{\mu-1} \frac{\gamma_{r+1,l}}{\varpi(\varphi)} (\varpi(u_l) - \varpi(\varphi) D_\tau^{r+1} + |\varpi|)\, \varphi^r u_l$$

$$\in \bigoplus_{j=0}^{\mu-1} \mathrm{Cas}(\mathcal{A}, \varphi)\, u_j.$$

The writing of $H_0^\tau(\mathcal{A},\varphi)$ leads exactly to the conclusion that the constants $\gamma_{r,l}$ and $\delta_{i,k}$ are equal to zero, so that the sum in (5.12) is a direct one. □

In the next Chapter, where we study the formal deformations of the Poisson structures of the form $\{\cdot\,,\cdot\}_\varphi$, we will need to obtain an other **F**-basis of the second Poisson cohomology space $H^2(\mathcal{A},\varphi)$, which is isomorphic to the first Poisson homology space $H_1(\mathcal{A},\varphi)$. We now will give an other **F**-basis of the space $H_1^\tau(\mathcal{A},\varphi)$, which will be analogous to the second **F**-basis of $H^2(\mathcal{A},\varphi)$, that will appear in Proposition 6.10.

Proposition 5.8. *Suppose $\tau \in \mathbf{F}\setminus\{-1,-2,-3,-4\}$. The MP-homology space $H_1^\tau(\mathcal{A},\varphi)$, which has been computed in Proposition 5.6, can also be written as:*

$$H_1^\tau(\mathcal{A},\varphi) \simeq \bigoplus_{r\in\mathbf{N}} \bigoplus_{j\in\mathcal{E}_\varphi^\tau(r)} \mathbf{F}\, \varphi^r u_j \vec{\nabla}\varphi \;\oplus\; \bigoplus_{k=1}^{\mu-1} \mathbf{F}\, \vec{\nabla} u_k, \tag{5.14}$$

where q_τ denotes $q_\tau := \dfrac{1}{\tau+2} \in \mathbf{F}$ and where

$$\mathcal{E}_\varphi^\tau(r) := \begin{cases} \{0,\dots,\mu-1\}, & \textit{if } \varpi(\varphi) = \left(\dfrac{1}{(r+1)\,q_\tau+1}\right)|\varpi|, \\ \{1,\dots,\mu-1\}, & \textit{otherwise.} \end{cases}$$

Remark 5.9. Notice that, for $\tau\in\mathbf{F}$, if there exists $r_0\in\mathbf{N}$, satifying the equality

$$\varpi(\varphi) = \left(\frac{1}{(r_0+1)\,q_\tau+1}\right)|\varpi|,$$

then r_0 is unique, i.e., for any $r\in\mathbf{N}\setminus\{r_0\}$, then $\varpi(\varphi) \neq \left(\dfrac{1}{(r+1)\,q_\tau+1}\right)|\varpi|$.

According to Proposition 5.8, we then can also write the space $H_1^\tau(\mathcal{A},\varphi)$, as follows

$$H_1^\tau(\mathcal{A},\varphi) \simeq \bigoplus_{j=1}^{\mu-1} \mathrm{Cas}(\mathcal{A},\varphi)\, u_j \vec{\nabla}\varphi \,\oplus\, \bigoplus_{k=1}^{\mu-1} \mathbf{F}\vec{\nabla} u_k$$
$$\oplus \begin{cases} \mathbf{F}\varphi^{r_0}\vec{\nabla}\varphi, & \text{if } \varpi(\varphi) = \left(\dfrac{1}{(r_0+1)\,q_\tau + 1}\right)|\varpi|, \\ \{0\}, & \text{if } \varpi(\varphi) \neq \left(\dfrac{1}{(r+1)\,q_\tau+1}\right)|\varpi|, \text{ for any } r \in \mathbf{N}. \end{cases}$$

Moreover, it is clear that if τ tends to infinity in the condition $\varpi(\varphi) = \dfrac{1}{((r+1)q_\tau+1)}\,|\varpi|$, we obtain the condition $\varpi(\varphi) = |\varpi|$ that is independent of r, so that, we will see in Proposition 6.10 that Proposition 5.8 holds for $\tau \to \infty$ (if we see the operator ∂_k^∞ as the classical operator, according to Remark 5.2).

Remark 5.10. By writing the equality obtained in (5.6), in the particular case of $j = 0$ (i.e. $u_j = 1$), we obtain:

$$\partial_2^\tau(\varphi^r \vec{e}_\varpi) = (-\varpi(\varphi)(r+\tau+3) + (\tau+2)|\varpi|)\, \varphi^r \vec{\nabla}\varphi \tag{5.15}$$

and

$$\text{if } \varpi(\varphi) \neq \frac{1}{((r+1)q_\tau+1)}\,|\varpi|, \text{ then } \varphi^r\vec{\nabla}\varphi \in B_1^\tau(\mathcal{A},\varphi).$$

Finally, notice that this condition is satisfied as soon as τ is irrational, as

$$\varpi(\varphi) = \frac{1}{((r+1)q_\tau+1)}\,|\varpi| \quad\Longleftrightarrow\quad \tau = \frac{\varpi(\varphi)(r+1)}{|\varpi| - \varpi(\varphi)} - 2.$$

Proof. The beginning of this proof is exactly the beginning of the proof of Proposition 5.6. Let us suppose that $\tau \in \mathbf{F}\backslash\{-1,-2,-3,-4\}$ and let $\vec{F} \in \mathcal{A}^3$, satisfying the 1-cycle condition: $\vec{\nabla}\varphi \cdot (\vec{\nabla} \times \vec{F}) = 0$. Then, according to Corollary 3.9, we have that there exist $G, H \in \mathcal{A}$ such that

$$\vec{F} = \vec{\nabla}G + H\vec{\nabla}\varphi. \tag{5.16}$$

Moreover, the writing of $H_0^\tau(\mathcal{A},\varphi)$ permits us to write (with $\vec{L},\vec{K} \in \mathcal{A}^3$):

$$G \in \partial_1^\tau(\vec{L}\,) + \sum_{j=1}^{\mu-1} \mathrm{Cas}(\mathcal{A},\varphi)u_j, \quad H \in \partial_1^\tau(\vec{K}\,) + \sum_{j=1}^{\mu-1} \mathrm{Cas}(\mathcal{A},\varphi)u_j.$$

We recall that, because $\tau \notin \{-2,-3\}$, the two elements $\vec{\nabla}(\delta_1^\varphi(\vec{L}\,))$ and $\delta_1^\varphi(\vec{K}\,)\,\vec{\nabla}\varphi$ are in $B_1^\tau(\mathcal{A},\varphi)$, as we have:

$$\vec{\nabla}(\delta_1^\varphi(\vec{L}\,)) = (\tau+1)\vec{\nabla}\left(\left(\vec{\nabla}\times\vec{L}\,\right)\cdot\vec{\nabla}\varphi\right) = -\frac{\tau+1}{\tau+3}\,\partial_2^\tau\left(\vec{\nabla}\times\vec{L}\,\right),$$

and:

$$\delta_1^{\varphi}(\vec{K}\ )\,\vec{\nabla}\varphi = (\tau+1)\left(\vec{\nabla}\varphi\cdot\left(\vec{\nabla}\times\vec{K}\ \right)\right)\,\vec{\nabla}\varphi = (\tau+1)\left(\mathrm{Div}\left(\vec{K}\times\vec{\nabla}\varphi\right)\right)\,\vec{\nabla}\varphi = \frac{\tau+1}{\tau+2}\,\partial_2^{\tau}\left(\vec{K}\times\vec{\nabla}\varphi\right).$$

Thus, we obtain:

$$\begin{aligned}\vec{F} &\in B_1^{\tau}(\mathcal{A},\varphi) + \sum_{j=1}^{\mu-1}\mathrm{Cas}(\mathcal{A},\varphi)\vec{\nabla}u_j + \sum_{j=0}^{\mu-1}\mathrm{Cas}(\mathcal{A},\varphi)u_j\vec{\nabla}\varphi \\ &\in B_1^{\tau}(\mathcal{A},\varphi) + \sum_{r\in\mathbf{N}}\sum_{j=1}^{\mu-1}\mathbf{F}\varphi^r\vec{\nabla}u_j + \sum_{r\in\mathbf{N}}\sum_{j=0}^{\mu-1}\mathbf{F}\varphi^r u_j\vec{\nabla}\varphi.\end{aligned} \tag{5.17}$$

Moreover, as $\tau \neq -3$, Lemma 5.5 allows us to say that, for $1 \leq j \leq \mu-1$ and for all $i \in \mathbf{N}$, we have $\varphi^{i+1}\vec{\nabla}u_j \in B_1^{\tau}(\mathcal{A},\varphi) + \mathbf{F}\ \varphi^i u_j\vec{\nabla}\varphi$, and, according to Formula (5.7), $\varphi^r\vec{\nabla}\varphi \in B_1^{\tau}(\mathcal{A},\varphi)$, as soon as $\varpi(\varphi) \neq |\varpi|/((r+1)q_\tau+1)$, i.e. as soon as $0 \notin \mathcal{E}_{\varphi}^{\tau}(r)$, so that (5.17) leads to:

$$Z_1^{\tau}(\mathcal{A},\varphi) \subseteq B_1^{\tau}(\mathcal{A},\varphi) + \sum_{j=1}^{\mu-1}\mathbf{F}\vec{\nabla}u_j + \sum_{r\in\mathbf{N}}\sum_{j\in\mathcal{E}_{\varphi}^{\tau}(r)}\mathbf{F}\varphi^r u_j\vec{\nabla}\varphi,$$

where $\mathcal{E}_{\varphi}^{\tau}(r)$ is defined above.

Now, we have to prove that this sum is a direct one. To do this, let us consider a weight homogeneous non zero element $\vec{H} \in \mathcal{A}^3 \simeq \Omega_d^2(\mathcal{A})$ and some elements $c_k, a_{r,j} \in \mathbf{F}$ for $1 \leq k \leq \mu-1$, $r \in \mathbf{N}$ and $j \in \mathcal{E}_{\varphi}^{\tau}(r)$ and let us suppose that we have the equation:

$$\partial_2^{\tau}(\vec{H}) + \sum_{k=1}^{\mu-1}c_k\,\vec{\nabla}u_k + \sum_{r\in\mathbf{N}}\sum_{j\in\mathcal{E}_{\varphi}^{\tau}(r)}a_{r,j}\ \varphi^r u_j\vec{\nabla}\varphi = 0, \tag{5.18}$$

with $\partial_2^{\tau}(\vec{H}) = -(\tau+3)\,\vec{\nabla}(\vec{H}\cdot\vec{\nabla}\varphi) + (\tau+2)\,\mathrm{Div}(\vec{H})\vec{\nabla}\varphi$. We take the inner product with $\vec{e}_{\varpi}$, in the equation (5.18) and the Euler Formula (3.5) leads to:

$$\sum_{k=1}^{\mu-1}c_k\varpi(u_k)\,u_k \in \langle\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial y},\frac{\partial\varphi}{\partial z}\rangle.$$

According to the definition of the u_k, this fact leads to $c_k = 0$, for all k between 1 and $\mu-1$. Then, we have obtained:

$$-(\tau+3)\,\vec{\nabla}(\vec{H}\cdot\vec{\nabla}\varphi) + (\tau+2)\,\mathrm{Div}(\vec{H})\vec{\nabla}\varphi + \sum_{r\in\mathbf{N}}\sum_{j\in\mathcal{E}_{\varphi}^{\tau}(r)}a_{r,j}\ \varphi^r u_j\vec{\nabla}\varphi = 0. \tag{5.19}$$

We have supposed that $\tau \neq -3$, so that this above equation leads to $\vec{\nabla}(\vec{H}\cdot\vec{\nabla}\varphi)\times\vec{\nabla}\varphi = 0$, that is to say, $\vec{H}\cdot\vec{\nabla}\varphi \in \mathrm{Cas}(\mathcal{A},\varphi)$ and this is a weight homogeneous element, so there exists $c \in \mathbf{F}$ and $i \in \mathbf{N}$, such that $\vec{H} \cdot \vec{\nabla}\varphi = c\varphi^{i+1}$. According to the exactness of the Koszul complex (Proposition 3.5), there exists a $\vec{G} \in \mathcal{A}^3$ such that:

$$\vec{H} = \frac{c}{\varpi(\varphi)}\varphi^i \vec{e}_\varpi + \vec{G} \times \vec{\nabla}\varphi,$$

and then, Equation (5.19) becomes:

$$\begin{aligned} -(\tau+3)c\,(i+1)\,\varphi^i\vec{\nabla}\varphi + (\tau+2)\Big(\frac{c}{\varpi(\varphi)}(i\varpi(\varphi)+|\varpi|)\Big)\varphi^i\vec{\nabla}\varphi \\ +(\tau+2)\left(\vec{\nabla}\varphi\cdot(\vec{\nabla}\times\vec{G})\right)\vec{\nabla}\varphi + \sum_{r\in\mathbf{N}}\sum_{j\in\mathcal{E}^\tau_\varphi(r)} a_{r,j}\;\varphi^r u_j\vec{\nabla}\varphi = 0, \end{aligned}$$

or, equivalently:

$$\begin{aligned} \frac{c}{\varpi(\varphi)}\Big(-\varpi(\varphi)(i+\tau+3)+(\tau+2)|\varpi|\Big)\varphi^i \\ +(\tau+2)\,\vec{\nabla}\varphi\cdot(\vec{\nabla}\times\vec{G}) + \sum_{r\in\mathbf{N}}\sum_{j\in\mathcal{E}^\tau_\varphi(r)} a_{r,j}\;\varphi^r u_j = 0, \end{aligned}$$

then, the the writing of $H_0^\tau(\mathcal{A},\varphi)$ $(\tau \neq -1)$ and of the $\mathcal{E}^\tau_\varphi(r)$ imply that the $a_{r,j}$ are equal to zero, that leads to the result desired. □

Now, we will study the second MP-homology space $H_2^\tau(\mathcal{A},\varphi)$. According to Proposition 3.23 and to Proposition 3.14, if $\varphi \in \mathcal{A} = \mathbf{F}[x,y,z]$ is a weight homogeneous polynomial with an isolated singularity, then the second classical Poisson homology space of the Poisson variety $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$ is given by:

$$H_2(\mathcal{A},\varphi) \simeq \begin{cases} \{0\} & \text{if } \varpi(\varphi) \neq |\varpi|; \\ \operatorname{Cas}(\mathcal{A},\varphi)\,\vec{e}_\varpi = \bigoplus_{i\in\mathbf{N}} \mathbf{F}\varphi^i\,\vec{e}_\varpi & \text{if } \varpi(\varphi) = |\varpi|. \end{cases}$$

In the following proposition, we give the second MP-homology space.

Proposition 5.11. *Let $\varphi \in \mathcal{A} = \mathbf{F}[x,y,z]$ be a weight homogeneous polynomial with an isolated singularity. We consider the Poisson variety $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$, where $\{\cdot\,,\cdot\}_\varphi$ is defined in (5.4).*

Assume that $\tau \in \mathbf{F}\setminus\{-1,-2,-3,-4\}$, then the second MP-homology space of the Poisson variety $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$ is then given by:

$$H_2^\tau(\mathcal{A},\varphi) \simeq \begin{cases} \{0\} & \text{if for any } r\in\mathbf{N},\ \varpi(\varphi) \neq \left(\dfrac{1}{(r+1)\,q_\tau+1}\right)|\varpi|, \\ \mathbf{F}\varphi^r\vec{e}_\varpi, & \text{if } r\in\mathbf{N} \text{ satisfies } \varpi(\varphi) = \left(\dfrac{1}{(r+1)\,q_\tau+1}\right)|\varpi|, \end{cases}$$

where we recall that q_τ denotes the element $q_\tau = \dfrac{1}{\tau+2} \in \mathbf{F}$.

Proof. This proof will also be adapted from the one for the determination of the classical first Poisson cohomology space, obtained in Proposition 3.14. We still assume that $\tau \in \mathbf{F} \setminus \{-1, -2, -3, -4\}$.

Let $\vec{F} \in \mathcal{A}^3 \simeq \Omega^2(\mathcal{A})$ be a weight homogeneous non zero element satisfying the 2-cycle condition:

$$-(\tau+3)\,\vec{\nabla}(\vec{F}\cdot\vec{\nabla}\varphi) + (\tau+2)\,\mathrm{Div}(\vec{F}\,)\vec{\nabla}\varphi = \vec{0}. \tag{5.20}$$

Then, as $\tau \neq -3$, $\vec{F}\cdot\vec{\nabla}\varphi$ is a weight homogeneous Casimir of the Poisson algebra $(\mathcal{A}, \{\cdot\,,\cdot\}_\varphi)$, that is to say, according to Proposition 3.11, there exist $c \in \mathbf{F}$ and $r \in \mathbf{N}$, such that

$$\vec{F}\cdot\vec{\nabla}\varphi = c\varphi^{r+1}. \tag{5.21}$$

If we denote by $\vec{G} \in \mathcal{A}^3$ the element defined by:

$$\vec{G} = \vec{F} - \frac{c}{\varpi(\varphi)}\varphi^r\vec{e}_\varpi,$$

then (5.21) leads to $\vec{G}\cdot\vec{\nabla}\varphi = 0$ and according to (5.20) and (5.15), we have:

$$\begin{aligned}\vec{0} = \partial_2^\tau(\vec{F}\,) &= \partial_2^\tau(\vec{G}\,) + \frac{c}{\varpi(\varphi)}\partial_2^\tau\,(\varphi^r\vec{e}_\varpi)\\ &= (\tau+2)\,\mathrm{Div}(\vec{G}\,)\vec{\nabla}\varphi + \frac{c}{\varpi(\varphi)}\Big(-\varpi(\varphi)(r+\tau+3) + (\tau+2)|\varpi|\Big)\,\varphi^r\vec{\nabla}\varphi,\end{aligned}$$

thus $\vec{G}$ satifies the equations:

$$\begin{cases}\vec{G}\cdot\vec{\nabla}\varphi = 0,\\ \mathrm{Div}(\vec{G}) = \dfrac{c}{\varpi(\varphi)}\left(\left(1+\dfrac{r+1}{\tau+2}\right)\varpi(\varphi) - |\varpi|\right)\varphi^r.\end{cases}$$

Using Lemma 3.13, we obtain that, necessarily,

$$c\left(\left(1+\frac{r+1}{\tau+2}\right)\varpi(\varphi) - |\varpi|\right) = 0. \tag{5.22}$$

Moreover, as $\vec{G}\cdot\vec{\nabla}\varphi = 0$, the exactness of the Koszul complex in Proposition 3.5 leads to the existence of a $\vec{H} \in \mathcal{A}^3$ satisfying $\vec{G} = \vec{H}\times\vec{\nabla}\varphi$ and then $0 = \mathrm{Div}(\vec{G}) = \vec{\nabla}\varphi\cdot(\vec{\nabla}\times\vec{H})$, so that, according to Corollary 3.9, there exists $G \in \mathcal{A}$ such that $\vec{G} = \vec{\nabla}G\times\vec{\nabla}\varphi$. Now, if

$$\left(1+\frac{r+1}{\tau+2}\right)\varpi(\varphi) - |\varpi| \neq 0,$$

according to Equation (5.22), $c = 0$ and $\vec{F} = \vec{\nabla} G \times \vec{\nabla}\varphi \in B_2(\mathcal{A}, \varphi)$. The second possibility is $\left(1 + \dfrac{r+1}{\tau+2}\right) \varpi(\varphi) - |\varpi| = 0$, and then $\vec{F} = c\varphi^r \vec{e}_\varpi + \vec{\nabla} G \times \vec{\nabla}\varphi \in \mathbf{F}\varphi^r \vec{e}_\varpi + B_2(\mathcal{A}, \varphi)$, where $r \in \mathbf{N}$ satisfies the Equation:

$$\left(1 + \frac{r+1}{\tau+2}\right) \varpi(\varphi) = |\varpi| \iff \varpi(\varphi) = \left(\frac{1}{(r+1)\, q_\tau + 1}\right) |\varpi|. \tag{5.23}$$

Then, it just remains to verify that, for $r \in \mathbf{N}$ satisfying the above equation (5.23), $\varphi^r \vec{e}_\varpi \notin B_2^\tau(\mathcal{A}, \varphi)$. Let $r \in \mathbf{N}$, satisfying (5.23) and let us suppose that there exist $c \in \mathbf{F}$ and $G \in \mathcal{A}$ such that:

$$c\,\varphi^r \vec{e}_\varpi = \vec{\nabla} G \times \vec{\nabla}\varphi.$$

Using Euler's Formula (3.5),

$$c\,\varpi(\varphi)\,\varphi^{r+1} = (c\,\varphi^r \vec{e}_\varpi) \cdot \vec{\nabla}\varphi = \left(\vec{\nabla} G \times \vec{\nabla}\varphi\right) \cdot \vec{\nabla}\varphi = 0,$$

that leads to $c = 0$. Then we have obtained the desired result. □

Remark 5.12. Notice that in the classical case, we have seen in Proposition 3.14 that:

$$H^1(\mathcal{A}, \varphi) \simeq H_2(\mathcal{A}, \varphi) \simeq \{0\},$$

as soon as $\varpi(\varphi) \neq |\varpi|$, where $|\varpi|$ is the sum of the weights of the variables x, y and z. The MP-homology permits to have a different result, namely, for any $\varphi \in \mathcal{A} = \mathbf{F}[x, y, z]$ and any $r \in \mathbf{N}$, if $\varpi(\varphi) \neq |\varpi|$, then one can chose a $\tau \in \mathbf{F}$, such that

$$H_2^\tau(\mathcal{A}, \varphi) \simeq \mathbf{F}\varphi^r \vec{e}_\varpi,$$

precisely, $\tau = \dfrac{(r+1)\varpi(\varphi)}{|\varpi| - \varpi(\varphi)} - 2$.

5.4 MP-homology for surfaces in $\mathbf{F}^3$

In this section, we want to determine the Mathieu-Poisson homology of Poisson surfaces in $\mathbf{F}^3$. Our two examples, as in Chapter 4, are: the affine space of dimension two, $\mathbf{F}^2$, and singular surfaces $\mathcal{F}_\varphi \subset \mathbf{F}^3$ given by the zeros of weight homogeneous polynomials $\varphi \in \mathcal{A} = \mathbf{F}[x, y, z]$ with an isolated singularity.

5.4.1 The MP-homology of $(\mathbf{F}^2, \{\cdot\,,\cdot\}^\psi)$

In this paragraph, we consider a polynomial $\psi \in \mathbf{F}[x, y]$ and the associated Poisson variety $(\mathbf{F}^2, \{\cdot\,,\cdot\}^\psi)$, where $\{\cdot\,,\cdot\}^\psi$ is the Poisson bracket defined in Paragraph 2.1.2, by

$$\{\cdot\,,\cdot\}^{\psi} = \psi\,\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

Let us consider the MP-homology complex of $(\mathbf{F}^2, \{\cdot\,,\cdot\}^{\psi})$. According to the definition (5.3), the MP-boundary operator $\partial_k^{\tau} : \Omega^k(\mathbf{F}^2) \to \Omega^{k-1}(\mathbf{F}^2)$ is given, for $F, G \in \mathcal{A}$, by:

$$\begin{aligned}
\partial_0^{\tau}(F) &= \tau\,\imath_{\pi}(\mathsf{d}f) = 0,\\
\partial_1^{\tau}(F\,\mathsf{d}x + G\,\mathsf{d}y) &= (\tau+1)\,\imath_{\pi} \circ \mathsf{d}(F\,\mathsf{d}x + G\,\mathsf{d}y) = (\tau+1)\,\partial_1(F\,\mathsf{d}x + G\,\mathsf{d}y)\\
\partial_2^{\tau}(F\,\mathsf{d}x \wedge \mathsf{d}y) &= -(\tau+3)\,\mathsf{d} \circ \imath_{\pi}(F\,\mathsf{d}x \wedge \mathsf{d}y) = -(\tau+3)\,\partial_2(F\,\mathsf{d}x \wedge \mathsf{d}y),
\end{aligned}$$

where $\partial_k = \imath_{\pi} \circ \mathsf{d} - \mathsf{d} \circ \imath_{\pi}$ is the classical Poisson boundary operator associated to $(\mathbf{F}^2, \{\cdot\,,\cdot\}^{\psi})$ (See (4.22)). So that, for generic values of τ, i.e., for $\tau \neq -1, -3$, the classical Poisson homology and the MP-homology of $(\mathbf{F}^2, \{\cdot\,,\cdot\}^{\psi})$ coincide:

$$H_k^{\tau}(\mathbf{F}^2, \{\cdot\,,\cdot\}^{\psi}) \simeq H_k(\mathbf{F}^2, \{\cdot\,,\cdot\}^{\psi}), \qquad \text{for } k = 0, 1, 2,$$

(See Paragraph 4.2.2 for the results of classical Poisson homology of the Poisson algebra $(\mathbf{F}^2, \{\cdot\,,\cdot\}^{\psi})$).

5.4.2 The MP-homology of the singular surface $\mathcal{F}_{\varphi}$

Let $\varphi \in \mathcal{A} = \mathbf{F}[x,y,z]$ be a weight homogeneous polynomial with an isolated singularity. We recall from Paragraph 4.3.1, that the singular surface $\mathcal{F}_{\varphi} : \{\varphi = 0\} \subseteq \mathbf{F}^3$ and its algebra of regular functions $\mathcal{A}_{\varphi} = \frac{\mathbf{F}[x,y,z]}{\langle\varphi\rangle}$ can be equipped with the Poisson bracket $\{\cdot\,,\cdot\}_{\mathcal{A}_{\varphi}}$, induced from $\{\cdot\,,\cdot\}_{\varphi}$, so that, modulo $\langle\varphi\rangle$, this Poisson bracket is given by

$$\{\cdot\,,\cdot\}_{\mathcal{A}_{\varphi}} = \frac{\partial\varphi}{\partial x}\,\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + \frac{\partial\varphi}{\partial y}\,\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + \frac{\partial\varphi}{\partial z}\,\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \pmod{\varphi}.$$

We want, in this paragraph to compute the MP-homology of the Poisson surface $(\mathcal{F}_{\varphi}, \{\cdot\,,\cdot\}_{\mathcal{A}_{\varphi}})$. To do this, we recall the identifications in the spaces of chains, given in Section 4.4.1. While $\Omega^0(\mathcal{A}_{\varphi}) \simeq \mathcal{A}_{\varphi}$, we have:

$$\begin{aligned}
\Omega^1(\mathcal{A}_{\varphi}) &\simeq \frac{\mathcal{A}_{\varphi}^3}{\{F\vec{\nabla}\varphi \mid F \in \mathcal{A}\}} \simeq \frac{\mathcal{A}^3}{\{F\vec{\nabla}\varphi + \varphi\vec{G} \mid F \in \mathcal{A},\, \vec{G} \in \mathcal{A}^3\}},\\
\Omega^2(\mathcal{A}_{\varphi}) &\simeq \frac{\mathcal{A}_{\varphi}^3}{\{\vec{\nabla}\varphi \times \vec{F} \mid \vec{F} \in \mathcal{A}^3\}} \simeq \frac{\mathcal{A}^3}{\{\vec{\nabla}\varphi \times \vec{F} + \varphi\vec{G} \mid \vec{F}, \vec{G} \in \mathcal{A}^3\}},\\
\Omega^3(\mathcal{A}_{\varphi}) &\simeq \frac{\mathcal{A}_{\varphi}}{\{\vec{\nabla}\varphi \cdot \vec{F} \mid \vec{F} \in \mathcal{A}^3\}} \simeq \frac{\mathcal{A}}{\langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z} \rangle} = \mathcal{A}_{sing}.
\end{aligned}$$

Moreover, the MP-homology operator $\partial_{k,\mathcal{A}_{\varphi}}^{\tau} : \Omega^k(\mathcal{A}_{\varphi}) \to \Omega^{k-1}(\mathcal{A}_{\varphi})$ is induced by the one ∂_k^{τ}, corresponding to the Poisson variety $(\mathbf{F}^3, \{\cdot\,,\cdot\}_{\varphi})$ and we obtain the MP-homology spaces of $\mathcal{A}_{\varphi}$:

$$H_0^\tau(\mathcal{A}_\varphi) \simeq \frac{\mathcal{A}}{\{(\tau+1)\vec{\nabla}\varphi \cdot (\vec{\nabla} \times \vec{F}) + \varphi G \mid G \in \mathcal{A}, \vec{F} \in \mathcal{A}^3\}},$$

$$H_1^\tau(\mathcal{A}_\varphi) \simeq$$
$$\frac{\{\vec{F} \in \mathcal{A}^3 \mid (\tau+1)\vec{\nabla}\varphi \cdot (\vec{\nabla} \times \vec{F}) \in \langle\varphi\rangle\}}{\{-(\tau+3)\vec{\nabla}(\vec{F}\cdot\vec{\nabla}\varphi) + (\tau+2)\operatorname{Div}(\vec{F})\,\vec{\nabla}\varphi + G\vec{\nabla}\varphi + \varphi\vec{H} \mid G \in \mathcal{A}, \vec{F}, \vec{H} \in \mathcal{A}^3\}},$$

$$H_2^\tau(\mathcal{A}_\varphi) \simeq \frac{\{\vec{F} \in \mathcal{A}^3 \mid -(\tau+3)\vec{\nabla}(\vec{F}\cdot\vec{\nabla}\varphi) + (\tau+2)\operatorname{Div}(\vec{F})\,\vec{\nabla}\varphi \in \mathcal{I}_\varphi\}}{\{\vec{\nabla}\varphi \times \vec{H} + \varphi\vec{K} \mid \vec{H}, \vec{K} \in \mathcal{A}^3\}},$$
$$\text{where } \mathcal{I}_\varphi := \{F\vec{\nabla}\varphi + \varphi\vec{G} \mid F \in \mathcal{A}, \vec{G} \in \mathcal{A}^3\},$$

$$H_3^\tau(\mathcal{A}_\varphi) \simeq \mathcal{A}_{sing} = \frac{F[x,y,z]}{\left\{\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right\}}.$$

We will denote by $Z_i^\tau(\mathcal{A}_\varphi)$ and $B_i^\tau(\mathcal{A}_\varphi)$ the spaces of all MP-cycles and MP-boundaries of $(\mathcal{A}_\varphi, \{\cdot,\cdot\}_{\mathcal{A}_\varphi})$. We point out that, for all parameter τ, we have $H_3^\tau(\mathcal{A}_\varphi) \simeq H_3(\mathcal{A}_\varphi) \simeq \mathcal{A}_{sing}$. Moreover, if $\tau \neq -1$, we have also $H_0^\tau(\mathcal{A}_\varphi) \simeq H_0(\mathcal{A}_\varphi) \simeq \mathcal{A}_{sing}$. The following proposition shows that, for all value of τ, except $-1, -2, -3$, the space $H_1^\tau(\mathcal{A}_\varphi)$ and, respectively $H_2^\tau(\mathcal{A}_\varphi)$, are isomorphic to $H_1(\mathcal{A}_\varphi)$ and, respectively, $H_2(\mathcal{A}_\varphi)$.

Proposition 5.13. *Let us suppose that $\tau \in \mathbf{F} \setminus \{-1,-2,-3\}$ and that $\varphi \in \mathcal{A} = \mathbf{F}[x,y,z]$ is a weight homogeneous polynomial, with an isolated singularity. Then, the first and the second MP-homology spaces of the Poisson surface $(\mathcal{F}_\varphi, \{\cdot,\cdot\}_{\mathcal{A}_\varphi})$ are given by the following:*

$$H_1^\tau(\mathcal{A}_\varphi) \simeq H_1(\mathcal{A}_\varphi) \simeq \bigoplus_{j=1}^{\mu-1} \mathbf{F}\vec{\nabla}u_j,$$
$$H_2^\tau(\mathcal{A}_\varphi) \simeq H_2(\mathcal{A}_\varphi) \simeq \bigoplus_{j=0}^{\mu-1} \mathbf{F}u_j\,\vec{e}_\varpi \simeq \mathcal{A}_{sing},$$

where, as in Paragraph 3.2.4,

$$\mathcal{A}_{sing} = \frac{F[x,y,z]}{\langle\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\rangle}$$

is the singular algebra associated to φ and $u_0 = 1, u_1, \ldots, u_{\mu-1} \in \mathcal{A}$ are weight homogeneous polynomials of $\mathbf{F}[x,y,z]$ whose images in $\mathcal{A}_{sing}$ give a $\mathbf{F}$-basis of this $\mathbf{F}$-vector space (See Paragraph 2.3.2 for the finite dimension).

Proof. We suppose $\tau \neq -1, -2, -3$.

• First, let us consider $H_1^\tau(\mathcal{A}_\varphi)$. We will follow the determination of $H_1(\mathcal{A}_\varphi)$ in Proposition 4.26 and adapt this case to the MP-one. Let $\vec{F} \in \mathcal{A}^3$ satisfying $(\tau+1)\vec{\nabla}\varphi \cdot (\vec{\nabla} \times \vec{F}) \in \langle \varphi \rangle$, thus, there exists $G \in \mathcal{A}$ with $\partial_1^\tau(\vec{F}\,) = (\tau+1)\vec{\nabla}\varphi \cdot (\vec{\nabla} \times \vec{F}) = \varphi G$.

As $\tau \neq -1$ and according to the expression of the space $H_0^\tau(\mathcal{A}, \varphi)$ (or $H^3(\mathcal{A}, \varphi)$ in Proposition 3.16), the sum

$$\mathcal{A} = \Big\{ (\tau+1)\, \vec{\nabla}\varphi \cdot (\vec{\nabla} \times \vec{F}\,) \mid \vec{F} \in \mathcal{A}^3 \Big\} \oplus \bigoplus_{j=0}^{\mu-1} \mathrm{Cas}(\mathcal{A}, \varphi)\, u_j$$

is a direct one, and both summands are stable by multiplication by φ, so that there exists $\vec{K} \in \mathcal{A}^3$ such that $G = (\tau+1)\vec{\nabla}\varphi \cdot (\vec{\nabla} \times \vec{K})$. Then $\vec{F} - \varphi\vec{K} \in Z_1^\tau(\mathcal{A}, \varphi)$ and, according to Proposition 5.8, we can write that:

$$\begin{aligned} \vec{F} - \varphi\vec{K} &\in \sum_{r \in \mathbf{N}} \sum_{j \in \mathcal{E}_\varphi^\tau(r)} \mathbf{F}\, \varphi^r u_j \vec{\nabla}\varphi + \sum_{k=1}^{\mu-1} \mathbf{F}\, \vec{\nabla} u_k + B_1^\tau(\mathcal{A}, \varphi) \\ &\in \sum_{k=1}^{\mu-1} \mathbf{F}\, \vec{\nabla} u_k + \{ \partial_2^\tau(\vec{L}\,) + J\vec{\nabla}\varphi + \varphi\vec{H} \mid J \in \mathcal{A},\ \vec{L}, \vec{H} \in \mathcal{A}^3 \}, \end{aligned}$$

so that

$$\vec{F} \in \sum_{k=1}^{\mu-1} \mathbf{F}\, \vec{\nabla} u_k + \{ \partial_2^\tau(\vec{L}\,) + J\vec{\nabla}\varphi + \varphi\vec{H} \mid J \in \mathcal{A},\ \vec{L}, \vec{H} \in \mathcal{A}^3 \}.$$

As, for all $0 \leq k \leq \mu - 1$, $\vec{\nabla} u_k \in Z_1^\tau(\mathcal{A}_\varphi)$, the family $\{\vec{\nabla} u_k,\ 0 \leq k \leq \mu - 1\}$ generates the $\mathbf{F}$-vector space $H_1^\tau(\mathcal{A}_\varphi)$. Let us show that this family is a $\mathbf{F}$-basis of $H_1^\tau(\mathcal{A}_\varphi)$. For this purpose, let us consider some elements $c_k \in \mathbf{F}$ (for $k = 1, \ldots, \mu - 1$), $J \in \mathcal{A}$, and $\vec{L}, \vec{H} \in \mathcal{A}^3$, satisfying:

$$\sum_{k=1}^{\mu-1} c_k \vec{\nabla} u_k = \partial_2^\tau(\vec{L}\,) + J\vec{\nabla}\varphi + \varphi\vec{H}.$$

We consider the inner product of the elements of this equation with the Euler derivation $\vec{e}_\varpi$, according to the Euler Formula (3.5), this operation leads to:

$$\sum_{k=1}^{\mu-1} c_k\, \varpi(u_k)\, u_k \in \langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z} \rangle,$$

and, according to the definition of the u_k, we have necessarily $c_k = 0$, for all $k = 1, \ldots, \mu - 1$, so that we have obtained:

$$H_1^\tau(\mathcal{A}_\varphi) \simeq \bigoplus_{k=1}^{\mu-1} \mathbf{F}\,\vec{\nabla} u_k.$$

• Now, let us consider the case of $H_2^\tau(\mathcal{A}_\varphi)$. One more time, we will follow the steps of the determination of $H_2(\mathcal{A}_\varphi)$ (Proposition 4.26) and adapt them to our MP-case.

Let $\vec{F} \in \mathcal{A}^3$ such that $\partial_2^\tau(\vec{F}) \in \mathcal{I}_\varphi$. There exist $L \in \mathcal{A}$, $\vec{G} \in \mathcal{A}^3$ satisfying

$$\partial_2^\tau(\vec{F}) = L\vec{\nabla}\varphi + \varphi\,\vec{G}. \tag{5.24}$$

We will study the two different terms on the right hand side of this equation.

1. First, let us work with $\varphi\,\vec{G}$. As $\vec{\nabla}\varphi \cdot \left(\vec{\nabla} \times (L\vec{\nabla}\varphi)\right) = 0$, we have $L\,\vec{\nabla}\varphi \in Z_1^\tau(\mathcal{A},\varphi)$ and $\varphi\,\vec{G} = \partial_2^\tau(\vec{F}) - L\vec{\nabla}\varphi \in Z_1^\tau(\mathcal{A},\varphi)$ and $\vec{G} \in Z_1^\tau(\mathcal{A},\varphi)$. Using the writing of $H_1^\tau(\mathcal{A},\varphi)$ in Proposition 5.8, we obtain the existence of $\vec{H} \in \mathcal{A}^3$, such that:

$$\vec{G} \in \sum_{r\in\mathbf{N}} \sum_{j\in\mathcal{E}_\varphi^\tau(r)} \mathbf{F}\,\varphi^r u_j \vec{\nabla}\varphi + \sum_{k=1}^{\mu-1} \mathbf{F}\,\vec{\nabla} u_k + \partial_2^\tau(\vec{H}),$$

where

$$\mathcal{E}_\varphi^\tau(r) := \begin{cases} \{0,\dots,\mu-1\}, \text{ if } \varpi(\varphi) = \left(\dfrac{1}{(r+1)\,q_\tau + 1}\right)|\varpi|, \\ \{1,\dots,\mu-1\}, \text{ otherwise.} \end{cases}$$

In order to study the element $\varphi\,\vec{G}$, let us observe the following computation.

$$\begin{aligned} \partial_2^\tau(\varphi\vec{H}) &= -(\tau+3)\vec{\nabla}(\varphi\vec{H}\cdot\vec{\nabla}\varphi) + (\tau+2)\,\mathrm{Div}(\varphi\vec{H})\,\vec{\nabla}\varphi \\ &= \varphi\left(-(\tau+3)\vec{\nabla}(\vec{H}\cdot\vec{\nabla}\varphi) + (\tau+2)\,\mathrm{Div}(\vec{H})\,\vec{\nabla}\varphi\right) \\ &\quad -(\tau+3)(\vec{H}\cdot\vec{\nabla}\varphi)\vec{\nabla}\varphi + (\tau+2)(\vec{\nabla}\varphi\cdot\vec{H})\,\vec{\nabla}\varphi \\ &= \varphi\partial_2^\tau(\vec{H}) - (\vec{H}\cdot\vec{\nabla}\varphi)\vec{\nabla}\varphi. \end{aligned}$$

This last equation permits us to obtain:

$$\varphi\,\vec{G} \in \sum_{r\in\mathbf{N}} \sum_{j\in\mathcal{E}_\varphi^\tau(r)} \mathbf{F}\,\varphi^{r+1} u_j \vec{\nabla}\varphi + \sum_{k=1}^{\mu-1} \mathbf{F}\,\varphi\vec{\nabla} u_k + \partial_2^\tau(\varphi\vec{H}) + (\vec{H}\cdot\vec{\nabla}\varphi)\vec{\nabla}\varphi.$$

Now, we have to decompose the different terms in the previous sum, as follows.

$$\begin{aligned} \sum_{r\in\mathbf{N}} \sum_{j\in\mathcal{E}_\varphi^\tau(r)} \mathbf{F}\,\varphi^{r+1} u_j \vec{\nabla}\varphi \subset & \sum_{r\in\mathbf{N}} \sum_{j\in\mathcal{E}_\varphi^\tau(r+1)} \mathbf{F}\,\varphi^{r+1} u_j \vec{\nabla}\varphi \\ & + \begin{cases} \mathbf{F}\varphi^{r_0+1}\vec{\nabla}\varphi, \text{ if } 0 \in \mathcal{E}_\varphi^\tau(r_0), \\ \{0\}, \qquad\qquad \text{otherwise.} \end{cases} \end{aligned}$$

because $\mathcal{E}_{\varphi}^{\tau}(r) \subset \mathcal{E}_{\varphi}^{\tau}(r+1) \cup \{0\}$ and if there exists $r_0 \in \mathbf{N}$ such that $0 \in \mathcal{E}_{\varphi}^{\tau}(r_0)$, it is unique. Moreover, according to Equation (5.6), we have:

$$\sum_{k=1}^{\mu-1} \mathbf{F}\, \varphi \vec{\nabla} u_k \subset \partial_2^{\tau}\left(\sum_{k=1}^{\mu-1} \mathbf{F} u_k \vec{e}_{\varpi}\right) + \sum_{k=1}^{\mu-1} \mathbf{F} u_k \vec{\nabla}\varphi,$$

and according to (5.15), if $0 \in \mathcal{E}_{\varphi}^{\tau}(r_0)$, then

$$\mathbf{F}\varphi^{r_0+1}\vec{\nabla}\varphi \subset \partial_2^{\tau}\left(\mathbf{F}\varphi^{r_0+1}\vec{e}_{\varpi}\right),$$

as, if $0 \in \mathcal{E}_{\varphi}^{\tau}(r_0)$, then $0 \notin \mathcal{E}_{\varphi}^{\tau}(r_0+1)$. Finally, we obtain:

$$\begin{aligned}
\varphi\,\vec{G} \in & \sum_{r\in\mathbf{N}} \sum_{j\in\mathcal{E}_{\varphi}^{\tau}(r+1)} \mathbf{F}\,\varphi^{r+1} u_j \vec{\nabla}\varphi + \sum_{k=1}^{\mu-1} \mathbf{F} u_k \vec{\nabla}\varphi + (\vec{H}\cdot\vec{\nabla}\varphi)\vec{\nabla}\varphi \\
& + \partial_2^{\tau}(\varphi\vec{H}) + \partial_2^{\tau}\left(\sum_{k=1}^{\mu-1} \mathbf{F} u_k \vec{e}_{\varpi}\right) + \partial_2^{\tau}\left(\sum_{\substack{r\in\mathbf{N}\\ 0\in\mathcal{E}_{\varphi}^{\tau}(r)}} \mathbf{F}\varphi^{r+1}\vec{e}_{\varpi}\right) \\
= & \sum_{r\in\mathbf{N}^*} \sum_{j\in\mathcal{E}_{\varphi}^{\tau}(r)} \mathbf{F}\,\varphi^{r} u_j \vec{\nabla}\varphi + \sum_{k=1}^{\mu-1} \mathbf{F} u_k \vec{\nabla}\varphi + (\vec{H}\cdot\vec{\nabla}\varphi)\vec{\nabla}\varphi \\
& + \partial_2^{\tau}\left(\varphi\vec{H} + \sum_{k=1}^{\mu-1} \mathbf{F} u_k \vec{e}_{\varpi} + \sum_{\substack{r\in\mathbf{N}\\ 0\in\mathcal{E}_{\varphi}^{\tau}(r)}} \mathbf{F}\varphi^{r+1}\vec{e}_{\varpi}\right) \\
\subset & \sum_{r\in\mathbf{N}} \sum_{j\in\mathcal{E}_{\varphi}^{\tau}(r)} \mathbf{F}\,\varphi^{r} u_j \vec{\nabla}\varphi + (\vec{H}\cdot\vec{\nabla}\varphi)\vec{\nabla}\varphi \\
& + \partial_2^{\tau}\left(\varphi\vec{H} + \sum_{k=1}^{\mu-1} \mathbf{F} u_k \vec{e}_{\varpi} + \sum_{\substack{r\in\mathbf{N}\\ 0\in\mathcal{E}_{\varphi}^{\tau}(r)}} \mathbf{F}\varphi^{r+1}\vec{e}_{\varpi}\right).
\end{aligned}$$

2. Now, let us consider the element $L'\vec{\nabla}\varphi$, where $L' = L + \vec{H}\cdot\vec{\nabla}\varphi$. According to the writing of $H_0^{\tau}(\mathcal{A})$, we have the existence of $\vec{K} \in \mathcal{A}^3$, such that:

$$L' \in \partial_1^{\tau}(\vec{K}) + \sum_{i\in\mathbf{N}} \sum_{j=0}^{\mu-1} \mathbf{F}\varphi^i u_j,$$

and then

$$L'\vec{\nabla}\varphi \in \partial_1^{\tau}(\vec{K})\vec{\nabla}\varphi + \sum_{i\in\mathbf{N}} \sum_{j=0}^{\mu-1} \mathbf{F}\varphi^i u_j \vec{\nabla}\varphi.$$

But, we have:

$$\begin{aligned}\partial_1^\tau(\vec{K})\vec{\nabla}\varphi &= (\tau+1)\left(\vec{\nabla}\varphi\cdot(\vec{\nabla}\times\vec{K}\,)\right)\vec{\nabla}\varphi\\ &= (\tau+1)\left(\mathrm{Div}(\vec{K}\times\vec{\nabla}\varphi)\right)\vec{\nabla}\varphi\\ &= \frac{\tau+1}{\tau+2}\quad\partial_2^\tau\left(\vec{K}\times\vec{\nabla}\varphi\right).\end{aligned}$$

Thus, we obtain:

$$\begin{aligned}L'\vec{\nabla}\varphi &\in \sum_{i\in\mathbf{N}}\sum_{j=0}^{\mu-1}\mathbf{F}\varphi^i u_j\vec{\nabla}\varphi + \partial_2^\tau\left(\frac{\tau+1}{\tau+2}\ \left(\vec{K}\times\vec{\nabla}\varphi\right)\right)\\ &\in \sum_{i\in\mathbf{N}}\sum_{j\in\mathcal{E}_\varphi^\tau(i)}\mathbf{F}\varphi^i u_j\vec{\nabla}\varphi + \sum_{\substack{i\in\mathbf{N}\\ 0\notin\mathcal{E}_\varphi^\tau(i)}}\mathbf{F}\varphi^i\vec{\nabla}\varphi + \partial_2^\tau\left(\frac{\tau+1}{\tau+2}\ \left(\vec{K}\times\vec{\nabla}\varphi\right)\right)\\ &\in \sum_{i\in\mathbf{N}}\sum_{j\in\mathcal{E}_\varphi^\tau(i)}\mathbf{F}\varphi^i u_j\vec{\nabla}\varphi + \partial_2^\tau\left(\sum_{\substack{i\in\mathbf{N}\\ 0\notin\mathcal{E}_\varphi^\tau(i)}}\mathbf{F}\varphi^i\vec{e}_\varpi\right) + \partial_2^\tau\left(\frac{\tau+1}{\tau+2}\ \left(\vec{K}\times\vec{\nabla}\varphi\right)\right),\end{aligned}$$

as, according to (5.7), if $0\notin\mathcal{E}_\varphi^\tau(i)$, then $\varphi^i\vec{\nabla}\varphi\in\partial_2^\tau(\mathbf{F}\varphi^i\vec{e}_\varpi)\in B_1^\tau(\mathcal{A})$.

Using the two precedent points, we can write the equation (5.24) as:

$$\begin{aligned}&\partial_2^\tau(\vec{F}) = \varphi\,\vec{G} + L\vec{\nabla}\varphi\\ &\in \sum_{r\in\mathbf{N}}\sum_{j\in\mathcal{E}_\varphi^\tau(r)}\mathbf{F}\,\varphi^r u_j\vec{\nabla}\varphi + \partial_2^\tau\left(\varphi\vec{H} + \sum_{k=1}^{\mu-1}\mathbf{F}u_k\vec{e}_\varpi + \sum_{\substack{r\in\mathbf{N}\\ 0\in\mathcal{E}_\varphi^\tau(r)}}\mathbf{F}\varphi^{r+1}\vec{e}_\varpi\right)\\ &\qquad +L'\vec{\nabla}\varphi\\ &\in \sum_{s\in\mathbf{N}}\sum_{l\in\mathcal{E}_\varphi^\tau(s)}\mathbf{F}\,\varphi^s u_l\vec{\nabla}\varphi + \partial_2^\tau\left(\varphi\vec{H} + \sum_{k=1}^{\mu-1}\mathbf{F}u_k\vec{e}_\varpi + \sum_{\substack{r\in\mathbf{N}\\ 0\in\mathcal{E}_\varphi^\tau(r)}}\mathbf{F}\varphi^{r+1}\vec{e}_\varpi\right)\\ &\quad +\partial_2^\tau\left(\sum_{\substack{i\in\mathbf{N}\\ 0\notin\mathcal{E}_\varphi^\tau(i)}}\mathbf{F}\varphi^i\vec{e}_\varpi\right) + \partial_2^\tau\left(\frac{\tau+1}{\tau+2}\ \left(\vec{K}\times\vec{\nabla}\varphi\right)\right).\end{aligned}$$

This equation implies that there exist some constants $\alpha_k,\beta_r,\gamma_i\in\mathbf{F}$, such that

$$\partial_2^\tau\Bigg(\vec{F}-\varphi\vec{H}-\sum_{k=1}^{\mu-1}\alpha_k u_k\vec{e}_\varpi-\sum_{\substack{r\in\mathbf{N}\\ 0\in\mathcal{E}_\varphi^\tau(r)}}\beta_r\varphi^{r+1}\vec{e}_\varpi-\sum_{\substack{i\in\mathbf{N}\\ 0\notin\mathcal{E}_\varphi^\tau(i)}}\gamma_i\varphi^i\vec{e}_\varpi \\ -\frac{\tau+1}{\tau+2}\left(\vec{K}\times\vec{\nabla}\varphi\right)\Bigg)\in\sum_{s\in\mathbf{N}}\sum_{l\in\mathcal{E}_\varphi^\tau(s)}\mathbf{F}\,\varphi^s u_l\vec{\nabla}\varphi \tag{5.25}$$

and, according to the writing of $H_1^\tau(\mathcal{A},\varphi)$ (Proposition 5.8), the following sum

$$\{\partial_2^\tau(\vec{F}\,)\mid\vec{F}\in\mathcal{A}^3\}\oplus\bigoplus_{s\in\mathbf{N}}\bigoplus_{l\in\mathcal{E}_\varphi^\tau(s)}\mathbf{F}\,\varphi^s u_l\vec{\nabla}\varphi$$

is a direct one, so that, the left hand side of (5.25) is equal to zero, that is to say:

$$\vec{F}-\varphi\vec{H}-\sum_{k=1}^{\mu-1}\alpha_k u_k\vec{e}_\varpi-\sum_{\substack{r\in\mathbf{N}\\ 0\in\mathcal{E}_\varphi^\tau(r)}}\beta_r\varphi^{r+1}\vec{e}_\varpi-\sum_{\substack{i\in\mathbf{N}\\ 0\notin\mathcal{E}_\varphi^\tau(i)}}\gamma_i\varphi^i\vec{e}_\varpi-\frac{\tau+1}{\tau+2}\left(\vec{K}\times\vec{\nabla}\varphi\right)$$
$$\in Z_2^\tau(\mathcal{A},\varphi)$$

$$=\{\vec{\nabla}F\times\vec{\nabla}\varphi\mid F\in\mathcal{A}\}$$
$$+\begin{cases}\{0\} & \text{if for any } t\in\mathbf{N}, \varpi(\varphi)\neq\left(\dfrac{1}{(t+1)\,q_\tau+1}\right)|\varpi|,\\ \mathbf{F}\varphi^{r_0}\vec{e}_\varpi, & \text{if } r_0\in\mathbf{N} \text{ satisfies } \varpi(\varphi)=\left(\dfrac{1}{(r_0+1)\,q_\tau+1}\right)|\varpi|.\end{cases}$$

So that, we obtain:

$$\vec{F}\in\sum_{k=1}^{\mu-1}\mathbf{F}u_k\vec{e}_\varpi+c_\varphi^\tau(0)\,\mathbf{F}\vec{e}_\varpi+\bar{c}_\varphi^\tau(0)\mathbf{F}\vec{e}_\varpi+\{\vec{\nabla}\varphi\times\vec{H}+\varphi\vec{K}\mid\vec{H},\vec{K}\in\mathcal{A}^3\},$$

with $c_\varphi^\tau(0),\bar{c}_\varphi^\tau(0)\in\mathbf{F}$ and $c_\varphi^\tau(0)\neq 0$ if and only if $0\notin\mathcal{E}_\varphi^\tau(0)$ while $\bar{c}_\varphi^\tau(0)\neq 0$ if and only if $\varpi(\varphi)=\left(\dfrac{1}{q_\tau+1}\right)|\varpi|$, i.e. $0\in\mathcal{E}_\varphi^\tau(0)$.

Finally, modulo $\{\vec{\nabla}\varphi\times\vec{H}+\varphi\vec{K}\mid\vec{H},\vec{K}\in\mathcal{A}^3\}$, we obtain exactly

$$\vec{F}\in\sum_{k=1}^{\mu-1}\mathbf{F}u_k\vec{e}_\varpi+\mathbf{F}\vec{e}_\varpi=\sum_{k=0}^{\mu-1}\mathbf{F}u_k\vec{e}_\varpi.$$

As, moreover, according to Equation (5.8), for all $0\leq k\leq\mu-1$,

$$\partial_2^\tau(u_k\vec{e}_\varpi)=\Big(-(\tau+3)\varpi(\varphi)+(\tau+2)(\varpi(u_k)+|\varpi|)\Big)u_k\vec{\nabla}\varphi$$
$$-(\tau+3)\varpi(\varphi)\varphi\vec{\nabla}u_k\qquad\in\mathcal{I}_\varphi,$$

that implies that the family $\{u_k\vec{e}_\varpi,\ 0 \le k \le \mu-1\}$ generates the $\mathbf{F}$-vector space $H_2^\tau(\mathcal{A}_\varphi)$:

$$H_2^\tau(\mathcal{A}_\varphi) \simeq \sum_{k=0}^{\mu-1} \mathbf{F} u_k \vec{e}_\varpi. \tag{5.26}$$

It remains to verify that the sum in (5.26) is a direct one. For this purpose, let us suppose that there exist some elements $c_k \in \mathbf{F}$, for $k = 0, \ldots, \mu-1$ and $\vec{H}, \vec{K} \in \mathcal{A}^3$, satisfying:

$$\sum_{k=0}^{\mu-1} c_k\ u_k \vec{e}_\varpi = \vec{\nabla}\varphi \times \vec{H} + \varphi \vec{K}.$$

The inner product with $\vec{\nabla}\varphi$ and Euler's Formula (3.5) in this equation lead to:

$$\sum_{k=0}^{\mu-1} c_k \varpi(\varphi)\ u_k \varphi = \varphi\ \vec{K} \cdot \vec{\nabla}\varphi,$$

i.e.

$$\sum_{k=0}^{\mu-1} c_k \varpi(\varphi)\ u_k = \vec{K} \cdot \vec{\nabla}\varphi \in \langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z} \rangle,$$

that means, according to the definition of the u_k, that $c_k = 0$, for all $k = 0, \ldots, \mu - 1$. $\square$

6

Application to deformation theory

In this chapter, we study the role played by the Poisson cohomology (and in particular, by the second and the third Poisson cohomology spaces) in the deformation theory and we use the determination of the Poisson cohomology we have obtained in the previous chapters to deform Poisson bracket in dimension three. We begin with an introduction of the deformation theory in the Poisson context. It is clear that one who knows the theory of deformation of Lie algebras or associative algebras will have an idea about the case of Poisson algebra. However, as it is difficult to find references about this subject and as we need to fix some notations and points of view, we recall with details the notions of deformation of Poisson structures, extensions of deformations and the link with the Poisson cohomology. To do this, we will be inspired by the paper [48] which concerns the associative case.

We will then apply this theory to the cases of the Poisson varieties $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$, $(\mathbf{F}^2, \{\cdot\,,\cdot\}^\psi)$, $(\mathcal{F}_\varphi, \{\cdot\,,\cdot\}_{\mathcal{A}_\varphi})$ and determine the deformations of the Poisson brackets $\{\cdot\,,\cdot\}_\varphi$ and $\{\cdot\,,\cdot\}_{\mathcal{A}_\varphi}$, when $\varphi \in \mathbf{F}[x,y,z]$ is a weight homogeneous polynomial with an isolated singularity, and the bracket $\{\cdot\,,\cdot\}^\psi$, when $\psi \in \mathbf{F}[x,y]$ is a square-free weight homogeneous polynomial.

6.1 Deformations of Poisson structures

Poisson cohomology appears naturally when one wants to deform, formally, the Poisson bracket on a Poisson variety. This means the following. For $(M, \{\cdot\,,\cdot\})$ a Poisson variety and its algebra of regular functions $\mathcal{A}$, we consider the ring $\mathcal{A}[[\nu]] = \mathcal{A} \otimes_{\mathbf{F}} \mathbf{F}[[\nu]]$ of formal power series in some indeterminate ν, with coefficients in $\mathcal{A}$; we also denote this ring by $\mathcal{A}^\nu$ and we write $\mathbf{F}^\nu$ for $\mathbf{F}[[\nu]]$. Thus, an element of $\mathcal{A}^\nu$ is of the form $\sum_{i\in\mathbf{N}} F_i\,\nu^i$, where all $F_i \in \mathcal{A}$. The associative commutative product "$\cdot$" on $\mathcal{A}$ extends naturally to a product on $\mathcal{A}^\nu$, which we still denote by "$\cdot$". Explicitly

$$\left(\sum_{i\in\mathbf{N}} F_i \nu^i\right) \cdot \left(\sum_{j\in\mathbf{N}} G_j \nu^j\right) = \sum_{i,j\in\mathbf{N}} F_i \cdot G_j\ \nu^{i+j},$$

which is well-defined, since for any $k \in \mathbf{N}$ the coefficient of ν^k in the product is a finite sum. The new product just amounts to extension of scalars, from $\mathbf{F}$ to $\mathbf{F}^\nu$: the new product is $\mathbf{F}^\nu$-linear and reduces to the old one when $\nu = 0$. For $n \in \mathbf{N}$, we will also need the algebra

$$\mathcal{A}_n^\nu := \mathcal{A}^\nu / \langle \nu^{n+1} \rangle,$$

which is obtained by taking the quotient of $\mathcal{A}^\nu$ by the ideal of $\mathcal{A}^\nu$ that is generated by ν^{n+1}. One computes in the algebra $(\mathcal{A}_n^\nu, \cdot)$ precisely like in $(\mathcal{A}^\nu, \cdot)$, putting $\nu^i = 0$ whenever $i > n$; it is an algebra over the ring $\mathbf{F}_n^\nu := \mathbf{F}^\nu / \langle \nu^{n+1} \rangle$. We have that $\mathcal{A}_0^\nu \simeq \mathcal{A}$, in a natural way; these algebras will be identified in the sequel, without further notice.

We could indeed do the same extension of scalars with the Poisson bracket $\{\cdot\,,\cdot\}$, but that is exactly what we do *not* want to do, rather we want to study all possible Poisson brackets on $(\mathcal{A}^\nu, \cdot)$ or on $(\mathcal{A}_n^\nu, \cdot)$ that specialize[1] to $\{\cdot\,,\cdot\}$ when $\nu = 0$.

6.1.1 Definition and equivalence

In order to make this precise, we first point out that every skew-symmetric $\mathbf{F}_n^\nu$-bilinear map $\varphi_{(n)} : \mathcal{A}_n^\nu \times \mathcal{A}_n^\nu \to \mathcal{A}_n^\nu$ leads to skew-symmetric $\mathbf{F}$-bilinear maps $\varphi_0, \dots, \varphi_n$ from $\mathcal{A}$ to $\mathcal{A}$ defined for $F, G \in \mathcal{A}$ by

$$\varphi_{(n)}(F,G) = \varphi_0(F,G) + \varphi_1(F,G)\nu + \cdots + \varphi_n(F,G)\nu^n, \tag{6.1}$$

and vice versa. Moreover, $\varphi_{(n)}$ is a biderivation of $\mathcal{A}_n^\nu$ if and only if each of the φ_i is a biderivation of $\mathcal{A}$. By a slight abuse of notation, we will also use the same notation for the $\mathbf{F}_n^\nu$-bilinear extensions of the maps φ_i to $\mathcal{A}_n^\nu$, which has the effect that (6.1) remains valid when F and G belong to $\mathbf{F}_n^\nu$.

Definition 6.1. *Let $(M, \{\cdot\,,\cdot\})$ be a Poisson variety and $\mathcal{A}$ be its algebra of regular functions. For $n \in \mathbf{N}$, a $\mathbf{F}_n^\nu$-linear skew-symmetric biderivation $\pi_{(n)}$ of $\mathcal{A}_n^\nu$,*

$$\pi_{(n)} = \pi_0 + \pi_1\nu + \cdots + \pi_n\nu^n, \tag{6.2}$$

is called an n-th order deformation *of $\{\cdot\,,\cdot\}$ if $\pi_{(n)}$ satisfies the Jacobi identity and if $\pi_0 = \{\cdot\,,\cdot\}$.*

For $m < n$ a skew-symmetric biderivation $\pi_{(n)}$ of $\mathcal{A}_n^\nu$ leads naturally to a skew-symmetric biderivation $\pi_{(m)}$ of $\mathcal{A}_m^\nu$: in (6.2), simply stop the sum at the term $\pi_m\nu^m$ and replace the base ring $\mathbf{F}_n^\nu$ by $\mathbf{F}_m^\nu$. In the case of an n-th order deformation of $\{\cdot\,,\cdot\}$ this leads to an m-th order deformation of $\{\cdot\,,\cdot\}$. Similarly, a $\mathbf{F}^\nu$-linear skew-symmetric biderivation π_* of $\mathcal{A}^\nu$ leads to a skew-symmetric biderivation π_m of $\mathcal{A}_m^\nu$, for any $m \in \mathbf{N}$. This leads to the following definition of a formal deformation of $\{\cdot\,,\cdot\}$.

[1] Physicists like to think of ν as a multiple of Planck's constant; then $\nu \to 0$ corresponds to taking the classical limit, i.e., the quantum Poisson brackets become classical Poisson brackets.

Definition 6.2. *Let* $(M, \{\cdot\,,\cdot\})$ *be a Poisson variety and* $\mathcal{A}$ *be its algebra of regular functions. An* $\mathbf{F}^\nu$*-linear skew-symmetric biderivation* π_* *of* $\mathcal{A}^\nu$ *is called a* formal deformation *of* $\{\cdot\,,\cdot\}$ *if for any* $n \in \mathbf{N}$*, the bracket* $\pi_{(n)}$*, restriction of* π_* *to* $\mathcal{A}^\nu_n$*, is an* n*-th order deformation of* $\{\cdot\,,\cdot\}$*.*

As we pointed out in paragraph 2.1.1, we could have defined Poisson algebras and also Poisson algebras over rings. If π_* is a formal deformation of $\{\cdot\,,\cdot\}$ then $(\mathcal{A}^\nu, \cdot, \pi_*)$ is a Poisson algebra over the ring $\mathbf{F}^\nu$ (and $\mathcal{A}^\nu$ is not an algebra of regular functions over an affine variety).

Together with the notion of deformation of Poisson structures, there is a notion of equivalence of deformations that is given in the following:

Definition 6.3. *Let* $(M, \{\cdot\,,\cdot\})$ *be a Poisson variety and* $\mathcal{A}$ *be its algebra of regular functions. One says that two formal deformations* π_* *and* π'_* *of* $\{\cdot\,,\cdot\}$ *are* equivalent *if there exists a* $\mathbf{F}^\nu$*-linear map* $\Phi : \mathcal{A}^\nu \to \mathcal{A}^\nu$*, satisfying:*

(1) $\Phi(F) = F \,(\mathrm{mod}\,\nu)$*, for all* $F \in \mathcal{A}$*;*
(2) $\Phi(FG) = \Phi(F)\Phi(G)$*, for all* $F, G \in \mathcal{A}$*;*
(3) $\Phi(\pi_*[F,G]) = \pi'_*[\Phi(F), \Phi(G)]$*, for all* $F, G \in \mathcal{A}$*.*

Similarly, one defines the notion of equivalence for n*-th order deformations, by replacing* $\mathbf{F}^\nu$ *with* $\mathbf{F}^\nu_n$ *and* $\mathcal{A}^\nu$ *with* $\mathcal{A}^\nu_n$*.*

Condition (1) implies that Φ is invertible; condition (2) implies that $\Phi : \mathcal{A}^\nu \to \mathcal{A}^\nu$ is a morphism of associative commutative algebras, then (3) means that Φ is a Poisson algebra isomorphism $(\mathcal{A}^\nu, \pi_*) \to (\mathcal{A}^\nu, \pi'_*)$. It can be expressed as the commutativity of the following diagram:

$$\begin{array}{ccc} \mathcal{A}^\nu \times \mathcal{A}^\nu & \xrightarrow{\pi_*} & \mathcal{A}^\nu \\ {\scriptstyle \Phi\times\Phi}\downarrow & & \downarrow{\scriptstyle \Phi} \\ \mathcal{A}^\nu \times \mathcal{A}^\nu & \xrightarrow{\pi'_*} & \mathcal{A}^\nu \end{array} \tag{6.3}$$

Remark 6.4. Let Φ be a $\mathbf{F}^\nu$-linear map $\Phi : \mathcal{A}^\nu \to \mathcal{A}^\nu$, satisfying Conditions (1) and (2), then Φ^{-1} is also morphism of associative commutative algebras. So that, if π_* is a formal deformation of $\{\cdot\,,\cdot\}$, then the following formula

$$P[F,G] := \Phi^{-1}\left(\pi_*[\Phi(F), \Phi(G)]\right), \quad F, G \in \mathcal{A}$$

defines a skew-symmetric biderivation P of $\mathcal{A}^\nu$ and so a formal deformation of $\{\cdot\,,\cdot\}$.

6.1.2 Link with Poisson cohomology: extensions of deformations

Since n-th order deformations of $\{\cdot\,,\cdot\}$ (now also denoted by π_0) restrict to m-th order deformations of $\{\cdot\,,\cdot\}$, for any $m < n$, it is naturally to approach the construction and the study of all (formal) deformations of $\{\cdot\,,\cdot\}$ by analyzing the extendability of an n-th order deformation to an $(n+1)$-th order deformation. Thus we suppose that $\pi_{(n)}$ is an n-order deformation of $\{\cdot\,,\cdot\} = \pi_0$,

$$\pi_{(n)}[F,G] = \pi_0[F,G] + \pi_1[F,G]\nu + \cdots + \pi_n[F,G]\nu^n,$$

for $F, G \in \mathcal{A}$. We wonder if there exists a skew-symmetric biderivation π_{n+1} of $\mathcal{A}$ such that

$$\pi_{(n+1)}[F,G] := \pi_0[F,G] + \pi_1[F,G]\nu + \cdots + \pi_n[F,G]\nu^n + \pi_{n+1}[F,G]\nu^{n+1}$$

defines an $(n+1)$-th order deformation of $\{\cdot\,,\cdot\}$, which we do by expressing that $\left[\pi_{(n+1)}, \pi_{(n+1)}\right]_S = 0$ (in $\mathcal{A}^\nu_{n+1}$). Since $\left[\pi_{(n)}, \pi_{(n)}\right]_S = 0$ (in $\mathcal{A}^\nu_n$), all that remains is

$$\sum_{\substack{i+j=n+1\\ i,j\geq 0}} [\pi_i, \pi_j]_S = 0,$$

which we write in terms of the Poisson coboundary operator as

$$\delta^2_{\pi_0}(\pi_{n+1}) = -[\pi_{n+1}, \pi_0]_S = \frac{1}{2} \sum_{\substack{i+j=n+1\\ i,j\geq 1}} [\pi_i, \pi_j]_S\,. \tag{6.4}$$

(In particular, for $n = 0$, we have $\delta^2_{\pi_0}(\pi_1) = 0$, i.e., $\pi_1 \in \mathfrak{X}^2(\mathcal{A})$ is a Poisson 2-cocycle of $(M, \{\cdot\,,\cdot\} = \pi_0)$.) This means that $\pi_{(n)}$ can be extended to a $(n+1)$-th deformation of $\{\cdot\,,\cdot\} = \pi_0$, if and only if the right hand side in (6.4) is a 3-coboundary; notice that it is a 3-cocycle, since

$$\begin{aligned}
\delta^3_{\pi_0}\left(\sum_{\substack{i+j=n+1\\ i,j\geq 1}} [\pi_i, \pi_j]_S\right) &= \sum_{\substack{i+j=n+1\\ i,j\geq 1}} \left[\pi_0, [\pi_i, \pi_j]_S\right]_S \\
&= -\sum_{\substack{i+j=n+1\\ i,j\geq 1}} \left(\left[\pi_i, [\pi_j, \pi_0]_S\right]_S + [\pi_j, [\pi_0, \pi_i]_S]_S\right) \\
&= 2\sum_{\substack{i+j=n+1\\ i,j\geq 1}} \left[\pi_i, \delta^2_{\pi_0}(\pi_j)\right]_S \\
&= \sum_{\substack{i+k+l=n+1\\ i,k,l\geq 1}} [\pi_i, [\pi_k, \pi_l]_S]_S \\
&= 0,
\end{aligned}$$

where we have used the graded Jacobi identity for $[\cdot\,,\cdot]_S$ to obtain the second and the last lines. It follows that the obstruction for extending a deformation of some order to the next order lies in the third Poisson cohomology space $H^3(\mathcal{A}, \pi_0)$. This shows the following proposition.

Proposition 6.5. *Let $(M, \{\cdot\,,\cdot\})$ be a Poisson variety. Suppose that $\pi_{(n)} = \sum_{i=0}^{n} \pi_i \nu^i$ is an n-th order deformation of $\pi_0 = \{\cdot\,,\cdot\}$. Then $\pi_{(n)}$ extends to an $(n{+}1)$-th order deformation if and only if the Poisson 3-cocycle $\sum_{i=1}^{n+1} [\pi_i, \pi_{n+1-i}]_S$ is a Poisson 3-coboundary of $(M, \{\cdot\,,\cdot\})$.*

Notice that, because the skew-symmetric 3-derivations of the Poisson surfaces $(\mathbf{F}^2, \{\cdot\,,\cdot\}^{\psi})$ and $(\mathcal{F}_\varphi, \{\cdot\,,\cdot\}_{\mathcal{A}_\varphi})$ are equal to zero, any n-th order deformation of the Poisson structure extends to a $(n+1)$-th order deformation in these cases (See Section 6.3).

It is clear that the term $\pi_{n+1}\nu^{n+1}$ that makes an n-th order deformation of $\{\cdot\,,\cdot\}$ into an $(n+1)$-th order deformation of $\{\cdot\,,\cdot\}$ is not unique (if it exists): one can always add a Poisson 2-cocycle. If this cocycle is a coboundary then the two extended $(n+1)$-th order deformations are equivalent. This is a particular case of the following proposition, that explains how equivalence of deformations of $\{\cdot\,,\cdot\}$ is measured by the second cohomology space of $(M, \{\cdot\,,\cdot\})$.

Proposition 6.6. *Let $(M, \{\cdot\,,\cdot\})$ be a Poisson variety and $\mathcal{A}$ be its algebra of regular functions. Let $\pi_{(n)} = \pi_0 + \pi_1\nu + \pi_2\nu^2 + \ldots + \pi_n\nu^n$ be an n-th order deformation of $\{\cdot\,,\cdot\}$, where $n \in \mathbf{N}^*$ and let $m \le n$, $m \in \mathbf{N}$. Let $\varphi \in \mathfrak{X}^1(\mathcal{A})$, which we extend to an $\mathbf{F}^\nu$-linear map $\mathcal{A}^\nu \to \mathcal{A}^\nu$, still denoted by φ, and let*

$$\pi'_{(m)} = \pi_0 + \pi_1\nu + \pi_2\nu^2 + \ldots + \pi_{m-1}\nu^{m-1} + (\pi_m + \delta^1_{\pi_0}(\varphi))\nu^m. \qquad (6.5)$$

Then, $\pi'_{(m)}$ extends to an n-th order deformation $\pi'_{(n)}$ of $\{\cdot\,,\cdot\}$, which is equivalent to $\pi_{(n)}$.

Proof. Let us consider the map

$$\begin{aligned} \varPhi : \mathcal{A}^\nu &\to \mathcal{A}^\nu \\ F &\mapsto \exp(-\nu^m\varphi)(F) \\ &\quad = F - \nu^m\varphi(F) + \tfrac{1}{2}\nu^{2m}\varphi^2(F) - \cdots \end{aligned}$$

The map $\varPhi$ is an $\mathbf{F}^\nu$-linear map that satisfies $\varPhi(F) = F \,(\mathrm{mod}\,\nu)$, for all $F \in \mathcal{A}$. Moreover, for all $F, G \in \mathcal{A}$, $\varPhi(FG) = \varPhi(F)\varPhi(G)$. Indeed, as φ is a derivation of $\mathcal{A}$, we have, for all $i \in \mathbf{N}$:

$$\varphi^i(FG) = \sum_{k=0}^{i} C_i^k\, \varphi^{i-k}(F)\, \varphi^k(G); \qquad (6.6)$$

So that, we obtain:

$$\begin{aligned} \varPhi(FG) = \exp(-\nu^m\varphi)(FG) &= \sum_{i\ge 0} \frac{(-1)^i}{i!}\nu^{im}\varphi^i(FG) \\ &= \sum_{i\ge 0}\sum_{k=0}^{i} \frac{(-1)^i}{i!}\, C_i^k\, \nu^{im}\, \varphi^{i-k}(F)\, \varphi^k(G) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i\geq 0}\sum_{k=0}^{i} \frac{(-1)^i}{k!(i-k)!}\,\nu^{im}\,\varphi^{i-k}(F)\,\varphi^k(G) \\
&= \sum_{j\geq 0}\sum_{k\geq 0} \frac{(-1)^{j+k}}{k!j!}\,\nu^{(j+k)m}\,\varphi^{j}(F)\,\varphi^k(G) \\
&= \left(\sum_{j\geq 0} \frac{(-1)^j}{j!}\nu^{jm}\varphi^j(F)\right)\left(\sum_{k\geq 0} \frac{(-1)^k}{k!}\nu^{km}\varphi^k(G)\right) \\
&= \Phi(F)\,\Phi(G).
\end{aligned}
$$

Then, according to Remark 6.4, the following formula

$$
\pi'_{(n)}[F,G] := \Phi(\pi_{(n)}[\Phi^{-1}(F),\Phi^{-1}(G)]),\ \text{for } F,G\in\mathcal{A}^\nu,
$$

defines a deformation of $\{\cdot\,,\cdot\}$ which is equivalent to $\pi_{(n)}$. The inverse of Φ is given, for $F\in\mathcal{A}$, by $\Phi^{-1}(F)=\exp(\nu^m\varphi)(F)$. We verify that $\pi'_{(m)}$, the restriction of $\pi'_{(n)}$ to an m-th order deformation, is indeed given by (6.5). Thus we compute, for $F,G\in\mathcal{A}$, in the algebra $\mathcal{A}^\nu_m$ ($\pi_{(m)}$ is the restriction of $\pi_{(n)}$ to an m-th order deformation):

$$
\begin{aligned}
\pi'_{(m)}[F,G] &= \Phi(\pi_{(m)}\,[F+\nu^m\varphi(F),G+\nu^m\varphi(G)]) \\
&= \Phi\left(\pi_{(m)}[F,G]+\nu^m(\pi_0[\varphi(F),G]+\pi_0[F,\varphi(G)])\right) \\
&= \pi_{(m)}[F,G]+\nu^m(\pi_0[\varphi(F),G]+\pi_0[F,\varphi(G)]-\varphi(\pi_0[F,G])) \\
&= \pi_{(m)}[F,G]+\nu^m(\delta^1_{\pi_0}\varphi)[F,G].
\end{aligned}
$$

□

Remark 6.7. We use the same notations than in Proposition 6.6. By observing the last computation in the proof of this proposition, we can be more precise about the deformation $\pi'_{(n)}$ obtained. We have indeed,

$$
\pi'_{(n)} = \pi_{(n)} + (\delta^1_{\pi_0}\varphi)\nu^m \mod \nu^{m+1},
$$

but in the proof of the next proposition, we will need to know what is the coefficient of ν^{m+1} in $\pi'_{(n)}$, if $m<n$. We will also denote by $\pi_{(k)}$ (respectively $\pi'_{(k)}$) the restriction of $\pi_{(n)}$ (respectively $\pi'_{(n)}$) to a k-th order deformation, when $k\leq n$.

First, suppose that $1<m<n$ ($m+1<2m$), and let us compute in the algebra $\mathcal{A}^\nu_{m+1}$,

$$
\begin{aligned}
\pi'_{(m+1)}(F,G) &= \Phi(\pi_{(m+1)}\,[F+\nu^m\varphi(F),G+\nu^m\varphi(G)]) \\
&= \Phi\big(\pi_{(m+1)}[F,G]+\nu^m(\pi_0[\varphi(F),G]+\pi_0[F,\varphi(G)]) \\
&\quad +\nu^{m+1}(\pi_1[\varphi(F),G]+\pi_1[F,\varphi(G)])\big) \\
&= \pi_{(m+1)}[F,G]+\nu^m(\pi_0[\varphi(F),G]+\pi_0[F,\varphi(G)]) \\
&\quad +\nu^{m+1}(\pi_1[\varphi(F),G]+\pi_1[F,\varphi(G)]) \\
&\quad -\nu^m(\varphi(\pi_0[F,G]))-\nu^{m+1}(\varphi(\pi_1[F,G])) \\
&= \pi_{(m+1)}[F,G]+\nu^m([\pi_0,\varphi]_S\,[F,G])+\nu^{m+1}([\pi_1,\varphi]_S\,[F,G]).
\end{aligned}
$$

So that, if we denote by π'_k the coefficient of ν^k in $\pi'_{(n)}$, we have, if $1 < m < n$,

$$\pi'_{m+1} = \pi_{m+1} + [\pi_1, \varphi]_S \,.$$

Now, let us suppose that $m = 1$ and $n > 1$. In this case, $\Phi(F) = F - \nu\varphi(F) + \frac{1}{2}\nu^2\varphi^2(F) \pmod{\nu^3}$ and $\Phi^{-1}(F) = F + \nu\varphi(F) + \frac{1}{2}\nu^2\varphi^2(F) \pmod{\nu^3}$, for all $F \in \mathcal{A}$. An analogous computation than in the previous case leads to:

$$\pi'_2 = \pi_2 + [\pi_1, \varphi]_S + \frac{1}{2}\left[\varphi, \delta^1_{\pi_0}(\varphi)\right]_S .$$

In the following proposition, we will use the particular fact that, in Proposition 6.6, we can chose $\pi'_{(n)}$ such that $\pi'_{m+1} - \pi_{m+1}$ depends only on φ and the π_j, with $j \leq m$.

It follows from Proposition 6.6 that in order to construct all possible deformations of $\{\cdot\,,\cdot\}$, up to equivalence, one only has to consider as many possibilities, at every step n, as there are elements in $H^2(\mathcal{A}, \pi_0)$.

6.1.3 A particular case

In this paragraph, we will give a result that can be applied only in very particular cases of Poisson structures, but, for example, in the case of $\{\cdot\,,\cdot\}_\varphi$, when φ is weight homogeneous, with an isolated singularity, as we will see in the next section.

Proposition 6.8. *Let $(M, \{\cdot\,,\cdot\} = \pi_0)$ be a Poisson variety and let $\mathcal{A}$ be its algebra of regular functions. Suppose that $(\vartheta_k, k \in \mathcal{K})$ is a family composed of 2-cocycles, whose images in $H^2(\mathcal{A}, \{\cdot\,,\cdot\})$ give an $\mathbf{F}$-basis of this $\mathbf{F}$-vector space. Suppose moreover that we have a family of formal deformations of the Poisson structure π_0, indexed by $\mathbf{a} = (a_k^n)_{\substack{k\in\mathcal{K}\\ n\in\mathbf{N}^*}}$, of the form:*

$$\pi_*^{\mathbf{a}} = \pi_0 + \sum_{n\in\mathbf{N}^*}\left(\Psi_n^{\bar{\mathbf{a}}_n} + \sum_{\substack{k\in\mathcal{K}\\ \text{finite}}} a_k^n\,\vartheta_k\right)\nu^n, \tag{6.7}$$

where $\Psi_n^{\bar{\mathbf{a}}_n} \in \mathfrak{X}^2(\mathcal{A}^\nu)$ is a family of skew-symmetric biderivations, indexed by $\bar{\mathbf{a}}_n = (a_k^m)_{\substack{k\in\mathcal{K}\\ 1\leq m<n}}$ and $\Psi_1^{\bar{\mathbf{a}}_1} = 0$.

- *Then, for any formal deformation π_* of π_0, there exists a sequence $\mathbf{b} = (b_k^n \in \mathbf{F})_{\substack{k\in\mathcal{K}\\ n\in\mathbf{N}^*}}$, such that π_* is equivalent to $\pi_*^{\mathbf{b}}$;*
- *Moreover, for any m-th order deformation $\pi_{(m)}$ of π_0 ($m \in \mathbf{N}^*$), there exists a sequence $\mathbf{b} = (b_k^n \in \mathbf{F})_{\substack{k\in\mathcal{K}\\ n\in\mathbf{N}^*}}$, such that $\pi_{(m)}$ is equivalent to $\pi_*^{\mathbf{b}}$ modulo ν^{m+1}, i.e., in $\mathcal{A}_m^\nu$.*

Proof. In the following, for $n \in \mathbf{N}^*$ and for a sequence $\mathbf{b} = (b_k^m \in \mathbf{F})_{\substack{k\in\mathcal{K}\\ m\in\mathbf{N}^*}}$, we will denote by $\overline{\mathbf{b}}_n$ and by $\underline{\mathbf{b}}_n$ the sequences defined by:

$$\overline{\mathbf{b}}_n := (b_k^m)_{\substack{k\in\mathcal{K}\\ 1\le m<n}}, \qquad \underline{\mathbf{b}}_n := (b_k^m)_{\substack{k\in\mathcal{K}\\ m\ge n}}.$$

Moreover, for two sequences $\overline{\mathbf{c}}_n = (c_k^m \in \mathbf{F})_{\substack{k\in\mathcal{K}\\ 1\le m<n}}$ and $\underline{\mathbf{c}}'_n = (c_k^{m\prime} \in \mathbf{F})_{\substack{k\in\mathcal{K}\\ m\ge n}}$, we define the concatenation of $\overline{\mathbf{c}}_n$ and $\underline{\mathbf{c}}'_n$, denoted by $(\overline{\mathbf{c}}_n, \underline{\mathbf{c}}'_n)$, as the sequence $(c_k^{m\prime\prime} \in \mathbf{F})_{\substack{k\in\mathcal{K}\\ m\in\mathbf{N}^*}}$, given by

$$\begin{aligned} c_k^{m\prime\prime} &= c_k^m, \quad \text{for all } k\in\mathcal{K},\ 1\le m<n,\\ c_k^{m\prime\prime} &= c_k^{m\prime}, \quad \text{for all } k\in\mathcal{K},\ m\ge n. \end{aligned}$$

So that, the concatenation of $\overline{\mathbf{b}}_n$ and $\underline{\mathbf{b}}_n$, $(\overline{\mathbf{b}}_n, \underline{\mathbf{b}}_n)$ is exactly $\mathbf{b}$.

Let us consider an arbitrary formal deformation of π_0, denoted by:

$$\pi_* = \pi_0 + \sum_{n\in\mathbf{N}^*} \pi_n \nu^n.$$

Let us construct a sequence $\mathbf{b} = (b_k^m \in \mathbf{F})_{\substack{k\in\mathcal{K}\\ m\in\mathbf{N}^*}}$, such that, for all $N \in \mathbf{N}$ and for all $\underline{\mathbf{c}}_{N+1} = (c_k^m \in \mathbf{F})_{\substack{k\in\mathcal{K}\\ m\ge N+1}}$, π_* is equivalent to $\pi_*^{\mathbf{b}_N}$ in $\mathcal{A}_N^\nu$, where $\mathbf{b}_N = (\overline{\mathbf{b}}_{N+1}, \underline{\mathbf{c}}_{N+1})$ is the concatenation of $\overline{\mathbf{b}}_{N+1}$ and $\underline{\mathbf{c}}_{N+1}$. Then, we will have obtained that π_* is equivalent to the deformation $\pi_*^{\mathbf{b}}$.

Moreover, if we replace π_* by a m-th order deformation $\pi_{(m)}$:

$$\pi_{(m)} = \pi_0 + \sum_{n=1}^{m} \pi_n \nu^n,$$

and if we do the same proof, but in $\mathcal{A}_m^\nu$ (and then, it suffices to consider $N \le m$), we will obtain, in the same way, that $\pi_{(m)}$ is equivalent to $\pi_*^{\mathbf{b}}$, modulo ν^{m+1}.

To do this, we will prove, by induction, that, for all $N \in \mathbf{N}^*$ (if we are proving the result concerning a m-th order deformation, it suffices to consider $N \le m$), there exist a sequence $\overline{\mathbf{b}}_{N+1} = (b_k^m \in \mathbf{F}, k\in\mathcal{K}, m\le N)$, a family of formal deformations $\pi_*^{N,\underline{\mathbf{c}}_{N+1}}$, indexed by $\underline{\mathbf{c}}_{N+1} = (c_k^m)_{\substack{k\in\mathcal{K}\\ m\ge N+1}}$, and a skew-symmetric biderivation $P_{N+1} \in \mathfrak{X}^2(\mathcal{A})$, depending on $\overline{\mathbf{b}}_{N+1}$, but not on $\underline{\mathbf{c}}_{N+1}$, such that:

$$\left.\begin{aligned} \pi_* &= \pi_*^{N,\underline{\mathbf{b}}_{N+1}} \pmod{\nu^{N+1}},\\ \pi_*^{N,\underline{\mathbf{b}}_{N+1}} &\simeq \pi_*^{\mathbf{b}},\\ \pi_{N+1}^{N,\underline{\mathbf{b}}_{N+1}} &= \pi_{N+1}^{\mathbf{b}} + P_{N+1}, \end{aligned}\right\} \begin{aligned} &\text{for all } \ \underline{\mathbf{b}}_{N+1} = (b_k^m \in \mathbf{F})_{\substack{k\in\mathcal{K}\\ m\ge N+1}},\\ &\text{denoting } \ \mathbf{b} := (\overline{\mathbf{b}}_N, \underline{\mathbf{b}}_{N+1}), \end{aligned}$$

where we denote by π_k (respectively $\pi_k^{N,\underline{\mathbf{c}}_{N+1}}$ and $\pi_k^{\mathbf{a}}$) the biderivation in factor of ν^k in the deformation π_* (respectively $\pi_*^{N,\underline{\mathbf{c}}_{N+1}}$ and $\pi_*^{\mathbf{a}}$), for all $k \in \mathbf{N}$. In particular, for all N, $\pi_*^{N,\underline{\mathbf{b}}_{N+1}}$ extends $(\pi_* \mod \nu^{N+1})$ and is equivalent to $\pi_*^{\mathbf{b}}$.

First, let us consider the case $N = 1$. We have by definition

$$\delta^2_{\pi_0}(\pi_1) = 0,$$

that is to say, π_1 is a 2-cocycle and this fact implies that there exist a sequence $\overline{\mathbf{b}}_2 = (b^1_k \in \mathbf{F})_{k\in\mathcal{K}}$, with $k \in \mathcal{K}$ and $\psi_1 \in \mathfrak{X}^1(\mathcal{A})$, such that:

$$\pi_1 = \sum_{k\in\mathcal{K}} b^1_k\, \vartheta_k + \delta^1_{\pi_0}(\psi_1).$$

According to the Proposition 6.6, for any $\underline{\mathbf{b}}_2 = (b^m_k \in \mathbf{F})_{\substack{k\in\mathcal{K}\\ m\geq 2}}$,

$$\pi_0 + \pi_1\nu = \pi_0 + \left(\sum_{k\in\mathcal{K}} b^1_k\, \vartheta_k + \delta^1_{\pi_0}(\psi_1)\right)\nu$$

extends to a formal deformation $\pi_*^{1,\underline{\mathbf{b}}_2}$ of π_0, which is equivalent to $\pi_*^{\mathbf{b}}$, where $\mathbf{b} := (\overline{\mathbf{b}}_2, \underline{\mathbf{b}}_2)$, because

$$\pi_*^{\mathbf{b}} = \pi_0 + \left(\sum_{k\in\mathcal{K}} b^1_k\, \vartheta_k\right)\nu \pmod{\nu^2}.$$

According to Remark 6.7, the skew-symmetric biderivation $P_2 := \pi_2 - \pi_2^{1,\underline{\mathbf{b}}_2}$ depends only on π_0, π_1, ψ_1 and not on $\underline{\mathbf{b}}_2$ and we then have obtained the desired result for $N = 1$.

Let now $N \geq 2$ be an integer and let us assume that there exist a sequence $\overline{\mathbf{b}}_{N+1} = (b^m_k \in \mathbf{F}, k \in \mathcal{K}, m \leq N)$, a family of formal deformations $\pi_*^{N,\underline{\mathbf{c}}_{N+1}}$, indexed by $\underline{\mathbf{c}}_{N+1} = (c^m_k)_{\substack{k\in\mathcal{K}\\ m\geq N+1}}$, and a skew-symmetric biderivation $P_{N+1} \in \mathfrak{X}^2(\mathcal{A})$, such that:

$$\left.\begin{aligned} \pi_* &= \pi_*^{N,\underline{\mathbf{d}}_{N+1}} \pmod{\nu^{N+1}},\\ \pi_*^{N,\underline{\mathbf{d}}_{N+1}} &\simeq \pi_*^{\mathbf{d}},\\ \pi_{N+1}^{N,\underline{\mathbf{d}}_{N+1}} &= \pi_{N+1}^{\mathbf{d}} + P_{N+1}, \end{aligned}\right\} \begin{array}{l} \text{for all } \underline{\mathbf{d}}_{N+1} = (d^m_k \in \mathbf{F})_{\substack{k\in\mathcal{K}\\ m\geq N+1}},\\ \text{denoting } \mathbf{d} := (\overline{\mathbf{b}}_{N+1}, \underline{\mathbf{d}}_{N+1}), \end{array}$$

We will prove the same for the rank $N + 1$. Let us first point out that, for all $\underline{\mathbf{d}}_{N+1}$,

$$\delta^2_{\pi_0}(\pi_{N+1}) = \delta^2_{\pi_0}\left(\pi_{N+1}^{N,\underline{\mathbf{d}}_{N+1}}\right),$$

because, according to Equation (6.4), this skew-symmetric 3-derivation is completely determined by the $\pi_j = \pi_j^{N,\underline{\mathbf{d}}_{N+1}}$, for $0 \leq j \leq N$. So that, there exist some constants $(b^{N+1}_k \in \mathbf{F})_{k\in\mathcal{K}}$ and a derivation $\psi_{N+1} \in \mathfrak{X}^1(\mathcal{A})$, satisfying

$$\pi_{N+1} = \pi_{N+1}^{N,\underline{\mathbf{d}}_{N+1}} + \sum_{k\in\mathcal{K}} b^{N+1}_k\, \vartheta_k + \delta^1_{\pi_0}(\psi_{N+1}).$$

We can write, using the third equation of the induction hypothese, for all $\underline{\mathbf{d}}_{N+1}$,

$$\begin{aligned}\pi_{N+1} &= \pi_{N+1}^{N,\underline{\mathbf{d}}_{N+1}} + \sum_{k\in\mathcal{K}} b_k^{N+1}\,\vartheta_k + \delta^1_{\pi_0}(\psi_{N+1})\\ &= \pi_{N+1}^{\mathbf{d}} + P_{N+1} + \sum_{k\in\mathcal{K}} b_k^{N+1}\,\vartheta_k + \delta^1_{\pi_0}(\psi_{N+1})\\ &= \pi_{N+1}^{\mathbf{D}} + P_{N+1} + \delta^1_{\pi_0}(\psi_{N+1}),\end{aligned}$$

where $\mathbf{D}$ is the sequence $\mathbf{D} = (D_k^n \in \mathbf{F})_{\substack{k\in\mathcal{K}\\ n\in\mathbf{N}^*}}$, defined by

$$\begin{aligned} D_k^n &= b_k^n, &&\text{for all } k\in\mathcal{K},\ 1\le n\le N+1,\\ D_k^n &= d_k^n, &&\text{for all } k\in\mathcal{K},\ n\ge N+2,\end{aligned}$$

i.e., $\mathbf{D}$ is the concatenation of the three sequences $\overline{\mathbf{b}}_{N+1}$, $(b_k^{N+1})_{k\in\mathcal{K}}$ and $\underline{\mathbf{d}}_{N+2}$. According to the third equality of the induction hypothese, we also have

$$\begin{aligned}\pi_{N+1} &= \pi_{N+1}^{\mathbf{D}} + P_{N+1} + \delta^1_{\pi_0}(\psi_{N+1})\\ &= \pi_{N+1}^{N,\underline{\mathbf{d}}_{N+2}} + \delta^1_{\pi_0}(\psi_{N+1}).\end{aligned}$$

Then, we can use one more time Proposition (6.6) to obtain, for all $\underline{\mathbf{d}}_{N+2}$, the existence of a formal deformation $\pi_*^{N+1,\underline{\mathbf{d}}_{N+2}}$ that extends

$$\pi_0 + \pi_1\nu + \cdots + \pi_{N+1}\nu^{N+1} = \pi_0 + \pi_1\nu + \cdots + \left(\pi_{N+1}^{N,\underline{\mathbf{d}}_{N+2}} + \delta^1_{\pi_0}(\psi_{N+1})\right)\nu^{N+1}$$

and is equivalent to $\pi_*^{\mathbf{D}}$. According to Remark 6.7, the skew-symmetric biderivation $P_{N+2} := \pi_{N+2}^{N+1,\underline{\mathbf{d}}_{N+2}} - \pi_{N+2}$ depends only on π_0, π_1, ψ_{N+1} and not on $\underline{\mathbf{d}}_{N+2}$ and we thus have obtained the desired result, for all $N\in\mathbf{N}^*$. Now, it is clear that what we have proved that π_* is equivalent to $\pi_*^{\mathbf{b}}$, where $\mathbf{b}$ is the sequence, constructed by induction. □

Remark 6.9. Let $(M, \{\cdot\,,\cdot\} = \pi_0)$ be a Poisson variety satisfying the conditions of Proposition 6.8. Then, $\{\cdot\,,\cdot\}$ satisfies the following property: for all $m\in\mathbf{N}^*$, every m-th order deformation of $\{\cdot\,,\cdot\}$ extends to a $(m+1)$-th order deformation of $\{\cdot\,,\cdot\}$.

Indeed, let $\pi_{(m)}$ be a m-th order deformation of $\{\cdot\,,\cdot\}$. Proposition 6.8 says, in particular, that there exists a sequence $\mathbf{b} = (b_k^n \in \mathbf{F})_{\substack{k\in\mathcal{K}\\ n\in\mathbf{N}^*}}$, such that $\pi_{(m)}$ is equivalent to $\pi_*^{\mathbf{b}}$ modulo ν^{m+1}, i.e., in $\mathcal{A}_m^\nu$. This leads to the existence of a $\mathbf{F}_m^\nu$-linear map $\Phi : \mathcal{A}_m^\nu \to \mathcal{A}_m^\nu$ satisfying (See Definition 6.3)

$$\begin{aligned}&\Phi(F) = F\,(\operatorname{mod}\nu),\\ &\pi_{(m)}[F,G] = \Phi^{-1}\left(\pi_*^{\mathbf{b}}[\Phi(F),\Phi(G)]\right)\ \operatorname{mod}\nu^{m+1},\end{aligned}$$

for all $F, G\in\mathcal{A}$. Then, we can naturally see Φ as a map $\mathcal{A}_{m+1}^\nu \to \mathcal{A}_{m+1}^\nu$ and the $(m+1)$-th order deformation $\pi_{(m+1)}$ of $\{\cdot\,,\cdot\}$, defined by

$$\pi_{(m+1)}[F,G] := \Phi^{-1}\left(\pi_*^{\mathbf{b}}[\Phi(F),\Phi(G)]\right) \bmod \nu^{m+2},$$

for all $F, G \in \mathcal{A}$ is an extension of $\pi_{(m)}$, hence the conclusion that $\pi_{(m)}$ extends to a $(m+1)$-th order deformation of $\{\cdot\,,\cdot\}$.

Poisson structures of the form $\{\cdot\,,\cdot\}_\varphi$, with $\varphi \in \mathbf{F}[x,y,z]$, a weight homogeneous polynomial with an isolated singularity, satisfy this property and we will see, in Paragraph 6.2.2, that we will be able to apply Proposition 6.8 in these cases (See Proposition 6.11). In Section 6.3, we will deal with the Poisson surfaces in $\mathbf{F}^3$: $(\mathbf{F}^2, \{\cdot\,,\cdot\}^\psi)$ and $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$ and point out that they also satisfy the hypotheses of Proposition 6.8 (See Propositions 6.13 and 6.14), but in an easier way.

6.2 Deformations in dimension three

In this paragraph, we determine the formal deformations of the Poisson structures of the form $\{\cdot\,,\cdot\}_\varphi$ on $\mathcal{A} = \mathbf{F}[x,y,z]$, where $\varphi \in \mathcal{A}$ is supposed to be weight homogeneous with an isolated singularity. As we have seen, the second Poisson cohomology space $H^2(\mathcal{A},\varphi)$ play an important role in this study. We first need to find a $\mathbf{F}$-basis of $H^2(\mathcal{A},\varphi)$ that is different from the one we have obtained in paragraph 3.2.5 and more adapted to the computations we do in this section. Then, we will see that Proposition 6.8 applies to the Poisson structures $\{\cdot\,,\cdot\}_\varphi$ and it will permit us to determine all the formal deformations of such Poisson structures, up to equivalence.

6.2.1 Another basis of $H^2(\mathcal{A},\varphi)$

As our purpose is to study the formal deformations of the Poisson structures $\{\cdot\,,\cdot\}_\varphi$ of $\mathbf{F}^3$ and because the second Poisson cohomology space will appear in this work, we need to have an appropriate expression of $H^2(\mathcal{A},\varphi)$.

Let $\mathcal{A} := \mathbf{F}[x,y,z]$ be the algebra of regular functions on the affine variety $\mathbf{F}^3$, equipped with the Poisson structure $\{\cdot\,,\cdot\}_\varphi$, where $\varphi \in \mathcal{A}$ is a weight-homogeneous polynomial with an isolated singularity. We recall that, explicitly, the Poisson bracket $\{\cdot\,,\cdot\}_\varphi$ is given by:

$$\{\cdot\,,\cdot\}_\varphi := \frac{\partial\varphi}{\partial x}\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z} + \frac{\partial\varphi}{\partial y}\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial x} + \frac{\partial\varphi}{\partial z}\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}. \tag{6.8}$$

Considering the Poisson algebra $(\mathcal{A}, \{\cdot\,,\cdot\}_\varphi)$, we have seen in Proposition 3.19 that

$$H^2(\mathcal{A},\varphi) \simeq \bigoplus_{\substack{j=1\\ \varpi(u_j)\neq\varpi(\varphi)-|\varpi|}}^{\mu-1} \mathrm{Cas}(\mathcal{A},\varphi)\vec{\nabla}u_j \oplus \bigoplus_{\substack{j=0\\ \varpi(u_j)=\varpi(\varphi)-|\varpi|}}^{\mu-1} \mathrm{Cas}(\mathcal{A},\varphi)\,u_j\vec{\nabla}\varphi$$
$$\oplus \bigoplus_{\substack{j=1\\ \varpi(u_j)=\varpi(\varphi)-|\varpi|}}^{\mu-1} \mathbf{F}\vec{\nabla}u_j, \tag{6.9}$$

where $\mathrm{Cas}(\mathcal{A},\varphi) \simeq \bigoplus_{i\in\mathbf{N}} \mathbf{F}\,\varphi^i$ and the family $u_j \in \mathcal{A}$, $j = 0,\dots,\mu-1$ is composed of weight-homogeneous polynomials whose images in

$$\mathcal{A}_{sing} = \mathbf{F}[x,y,z]/\langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\rangle$$

give a $\mathbf{F}$-basis of this $\mathbf{F}$-vector space (and $u_0 = 1$).

We first give another basis of $H^2(\mathcal{A},\varphi)$, which will be more useful for studying the formal deformations of the Poisson bracket $\{\cdot\,,\cdot\}_\varphi$.

Proposition 6.10. *If $\varphi \in \mathcal{A} = \mathbf{F}[x,y,z]$ is a weight homogeneous polynomial with an isolated singularity, then the second Poisson cohomology space of the Poisson algebra $(\mathcal{A},\{\cdot\,,\cdot\}_\varphi)$ is the* $\mathrm{Cas}(\mathcal{A},\varphi)$*-module:*

$$H^2(\mathcal{A},\varphi) \simeq \begin{cases} \displaystyle\bigoplus_{j=0}^{\mu-1} \mathrm{Cas}(\mathcal{A},\varphi)\, u_j\vec{\nabla}\varphi \;\oplus\; \bigoplus_{j=1}^{\mu-1} \mathbf{F}\,\vec{\nabla}u_j, & \text{if } \varpi(\varphi) = |\varpi|, \\ \\ \displaystyle\bigoplus_{j=1}^{\mu-1} \mathrm{Cas}(\mathcal{A},\varphi)\, u_j\vec{\nabla}\varphi \;\oplus\; \bigoplus_{j=1}^{\mu-1} \mathbf{F}\,\vec{\nabla}u_j, & \text{if } \varpi(\varphi) \neq |\varpi|, \end{cases}$$

where $\varpi(\varphi)$ is the (weighted) degree of φ, $|w|$ is the sum of the weights of the variables x, y and z and the u_j are weight homogeneous polynomials of $\mathcal{A}$, whose images in $\mathcal{A}_{sing} = \mathbf{F}[x,y,z]/\langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\rangle$ give a $\mathbf{F}$-basis of this $\mathbf{F}$-vector space, whose dimension is denoted by μ.

Proof. In order to simplify the proof, we define the set

$$\mathcal{E}_\varphi := \begin{cases} \{0,\dots,\mu-1\}, \text{ if } \varpi(\varphi) = |\varpi|, \\ \{1,\dots,\mu-1\}, \text{ if } \varpi(\varphi) \neq |\varpi|, \end{cases}$$

so that, we want to show that

$$H^2(\mathcal{A},\varphi) \simeq \bigoplus_{j\in\mathcal{E}_\varphi} \mathrm{Cas}(\mathcal{A},\varphi)\, u_j\vec{\nabla}\varphi \oplus \bigoplus_{j=1}^{\mu-1} \mathbf{F}\,\vec{\nabla}u_j.$$

We recall from Remark 3.21 that

$$\delta^1_\varphi\Big(\varphi^i u_j \vec{e}_\varpi\Big) = \Big(\varpi(u_j) - \varpi(\varphi) + |\varpi|\Big)\varphi^i u_j\vec{\nabla}\varphi - \varpi(\varphi)\,\varphi^{i+1}\vec{\nabla}u_j, \qquad (6.10)$$

for all $i \in \mathbf{N}$ and $j = 0,\dots,\mu-1$. We point out that this formula, specialized to case $j = 0$, leads immediatly to $\mathbf{F}\varphi^i\vec{\nabla}\varphi \subseteq B^2(\mathcal{A},\varphi)$, as soon as $\varpi(\varphi) \neq |\varpi|$ and $i \in \mathbf{N}$. This fact explains that we need use the set $\mathcal{E}_\varphi$ in the writing of $H^2(\mathcal{A},\varphi)$.

Formula (6.10) implies also that, for all $i \geq 1$,

$$\sum_{\substack{j=1\\ \varpi(u_j)\neq\varpi(\varphi)-|\varpi|}}^{\mu-1} \mathbf{F}\,\varphi^i\vec{\nabla}u_j \subseteq B^2(\mathcal{A},\varphi)+\sum_{j=1}^{\mu-1}\mathrm{Cas}(\mathcal{A},\varphi)\,u_j\vec{\nabla}\varphi$$
$$\subseteq B^2(\mathcal{A},\varphi)+\sum_{j\in\mathcal{E}_\varphi}\mathrm{Cas}(\mathcal{A},\varphi)\,u_j\vec{\nabla}\varphi.$$

Moreover, the definition of $\mathcal{E}_\varphi$ leads to

$$\sum_{\substack{j=0\\ \varpi(u_j)=\varpi(\varphi)-|\varpi|}}^{\mu-1} \mathrm{Cas}(\mathcal{A},\varphi)\,u_j\vec{\nabla}\varphi \subseteq \sum_{j\in\mathcal{E}_\varphi}\mathrm{Cas}(\mathcal{A},\varphi)\,u_j\vec{\nabla}\varphi,$$

so that, according to (6.9),

$$H^2(\mathcal{A},\varphi)\ \subseteq\ \sum_{j\in\mathcal{E}_\varphi}\mathrm{Cas}(\mathcal{A},\varphi)\,u_j\vec{\nabla}\varphi+\sum_{j=1}^{\mu-1}\mathbf{F}\,\vec{\nabla}u_j.$$

The other inclusion is clear, so we now have just to show that this sum is a direct one, modulo $B^2(\mathcal{A},\varphi)$. For this, let us consider some elements of $\mathbf{F}$, $\lambda_j,\delta_{i,k}\in\mathbf{F}$, for $i\in\mathbf{N}$, $1\leq j\leq\mu-1$, $k\in\mathcal{E}_\varphi$ and an element $\vec{H}\in\mathcal{A}^3$, satisfying the equation in $\mathfrak{X}^2(\mathcal{A})$:

$$\sum_{i\in\mathbf{N}}\sum_{k\in\mathcal{E}_\varphi}\delta_{i,k}\,\varphi^i u_k\,\vec{\nabla}\varphi+\sum_{j=1}^{\mu-1}\lambda_j\,\vec{\nabla}u_j=\delta^1_\varphi(\vec{H})=-\vec{\nabla}(\vec{H}\cdot\vec{\nabla}\varphi)+\mathrm{Div}(\vec{H})\vec{\nabla}\varphi,$$

where the right hand side of this equality is an element of $B^2(\mathcal{A},\varphi)$. We can write this equation as:

$$\vec{\nabla}\left(\sum_{j=1}^{\mu-1}\lambda_j\,u_j+\vec{H}\cdot\vec{\nabla}\varphi\right)=\left(\mathrm{Div}(\vec{H})-\sum_{i\in\mathbf{N}}\sum_{k\in\mathcal{E}_\varphi}\delta_{i,k}\,\varphi^i u_k\right)\vec{\nabla}\varphi,\qquad(6.11)$$

so that the element $\sum_{j=1}^{\mu-1}\lambda_j\,u_j+\vec{H}\cdot\vec{\nabla}\varphi\in\mathcal{A}$ satisfies the 0-cocycle condition, that is to say, is a Casimir for $(\mathcal{A},\{\cdot\,,\cdot\}_\varphi)$ and this fact implies, according to Proposition 3.11, that there exist elements $c_r\in\mathbf{F}$, with $r\geq 1$, such that:

$$\sum_{j=1}^{\mu-1}\lambda_j\,u_j+\vec{H}\cdot\vec{\nabla}\varphi=\sum_{r\in\mathbf{N}^*}c_r\,\varphi^r,\qquad(6.12)$$
$$\iff\quad \sum_{j=1}^{\mu-1}\lambda_j\,u_j=\sum_{r\in\mathbf{N}^*}c_r\,\varphi^r-\vec{H}\cdot\vec{\nabla}\varphi\in\langle\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial y},\frac{\partial\varphi}{\partial z}\rangle.$$

The definition of the u_j leads to $\lambda_j=0$, for all $j=1,\ldots,\mu-1$, so that we have:

$$\vec{H}\cdot\vec{\nabla}\varphi=\sum_{r\in\mathbf{N}^*}c_r\,\varphi^r=\sum_{r\in\mathbf{N}^*}\frac{c_r}{\varpi(\varphi)}\,\varphi^{r-1}\,\vec{e}_\varpi\cdot\vec{\nabla}\varphi,$$

where we use the Euler Formula (3.5), for the last equality.

Using the exactness of the Koszul complex (Proposition 3.5), we have the existence of an element $\vec{K}\in\mathcal{A}^3$, satisfying

$$\vec{H}=\sum_{r\in\mathbf{N}^*}\frac{c_r}{\varpi(\varphi)}\,\varphi^{r-1}\,\vec{e}_\varpi+\vec{K}\times\vec{\nabla}\varphi,$$

that gives

$$\mathrm{Div}(\vec{H})=\sum_{r\in\mathbf{N}^*}\frac{c_r}{\varpi(\varphi)}(\varpi(\varphi)(r-1)+|\varpi|)\,\varphi^{r-1}+(\vec{\nabla}\times\vec{K})\cdot\vec{\nabla}\varphi. \tag{6.13}$$

We consider (6.11), with the equations (6.12) and (6.13) to obtain:

$$\begin{aligned}\sum_{r\in\mathbf{N}^*}rc_r\varphi^{r-1}=&\sum_{r\in\mathbf{N}^*}\frac{c_r}{\varpi(\varphi)}\Big(\varpi(\varphi)(r-1)+|\varpi|\Big)\,\varphi^{r-1}+(\vec{\nabla}\times\vec{K})\cdot\vec{\nabla}\varphi\\&-\sum_{i\in\mathbf{N}}\sum_{k\in\mathcal{E}_\varphi}\delta_{i,k}\,\varphi^i u_k,\end{aligned}$$

after simplifying by $\vec{\nabla}\varphi$. That gives

$$\sum_{r\in\mathbf{N}^*}c_r\left(1-\frac{|\varpi|}{\varpi(\varphi)}\right)\varphi^{r-1}=(\vec{\nabla}\times\vec{K})\cdot\vec{\nabla}\varphi-\sum_{i\in\mathbf{N}}\sum_{k\in\mathcal{E}_\varphi}\delta_{i,k}\,\varphi^i u_k.$$

This equation means that, if $\varpi(\varphi)=|\varpi|$ (equivalently, $\mathcal{E}_\varphi=\{0,\dots,\mu-1\}$),

$$0=(\vec{\nabla}\times\vec{K})\cdot\vec{\nabla}\varphi-\sum_{i\in\mathbf{N}}\sum_{k=0}^{\mu-1}\delta_{i,k}\,\varphi^i u_k, \tag{6.14}$$

while, if $\varpi(\varphi)\neq|\varpi|$ ($\mathcal{E}_\varphi=\{1,\dots,\mu-1\}$),

$$\sum_{r\in\mathbf{N}^*}c_r\left(1-\frac{|\varpi|}{\varpi(\varphi)}\right)\varphi^{r-1}=(\vec{\nabla}\times\vec{K})\cdot\vec{\nabla}\varphi-\sum_{i\in\mathbf{N}}\sum_{k=1}^{\mu-1}\delta_{i,k}\,\varphi^i u_k. \tag{6.15}$$

Because of the computation of $H^3(\mathcal{A},\varphi)$ in Proposition 3.16, we have the direct sum

$$\mathcal{A}=\{(\vec{\nabla}\times\vec{F})\cdot\vec{\nabla}\varphi\mid\vec{F}\in\mathcal{A}^3\}\oplus\bigoplus_{\substack{i\in\mathbf{N}\\k=0,\dots,\mu-1}}\mathbf{F}\,\varphi^i u_k,$$

that permits us to conclude, using (6.14) and (6.15), that, in both cases, $\varpi(\varphi)=|\varpi|$ and $\varpi(\varphi)\neq|\varpi|$, we have $\delta_{i,k}=0$, for all $i\in\mathbf{N}$ and $k\in\mathcal{E}_\varphi$, so that the sum considered is an exact one, hence the result. □

6.2.2 The formal deformations of $\{\cdot\,,\cdot\}_\varphi$

In this part, our purpose is to consider the formal deformations of the Poisson bracket $\pi_0 := \{\cdot\,,\cdot\}_\varphi$ on $\mathbf{F}^3$, where φ is a weight homogeneous polynomial with an isolated singularity. For this work, the Poisson cohomology that appears is the one associated to the Poisson variety $(\mathbf{F}^3, \pi_0 = \{\cdot\,,\cdot\}_\varphi)$.

We first need to obtained a formula for the Schouten bracket of two specific skew-symmetric biderivations of $\mathcal{A}$. In fact, for the study of the formal deformations of $\{\cdot\,,\cdot\}_\varphi$, we will see that one only has to consider the skew-symmetric biderivations of the form $F\,\vec{\nabla}G \in \mathcal{A}^3 \simeq \mathfrak{X}^2(\mathcal{A})$, with $F, G \in \mathcal{A}$ (See Paragraph 3.1.1, for the identification $\mathcal{A}^3 \simeq \mathfrak{X}^2(\mathcal{A})$). Let us compute the Schouten bracket of such a skew-symmetric biderivation and an arbitrary skew-symmetric biderivation. So let $\vec{F} = (F_1, F_2, F_3) \in \mathcal{A}^3 \simeq \mathfrak{X}^2(\mathcal{A})$ and let $G, H \in \mathcal{A}$. We recall that, under the identifications in Paragraph 3.1.1, $\vec{F}$ and $G\vec{\nabla}H$ denote the skew-symmetric biderivations:

$$\begin{aligned}\vec{F} &= F_1\,\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + F_2\,\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + F_3\,\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y},\\ G\,\vec{\nabla}H &= G\,\frac{\partial H}{\partial x}\,\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + G\,\frac{\partial H}{\partial y}\,\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + G\,\frac{\partial H}{\partial z}\,\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.\end{aligned}$$

Then, with Definition (2.15), we compute the Schouten bracket $\left[\vec{F}, G\,\vec{\nabla}H\right]_S \in \mathfrak{X}^3(\mathcal{A}) \simeq \mathcal{A}$,

$$\begin{aligned}\left[\vec{F}, G\,\vec{\nabla}H\right]_S[x,y,z] &= \vec{F}\left[G\frac{\partial H}{\partial z}, z\right] + \vec{F}\left[G\frac{\partial H}{\partial x}, x\right] + \vec{F}\left[G\frac{\partial H}{\partial y}, y\right]\\ &\quad + G\,\vec{\nabla}H[F_3, z] + G\,\vec{\nabla}H[F_1, x] + G\,\vec{\nabla}H[F_2, y]\\ &= F_1\frac{\partial}{\partial y}\left(G\frac{\partial H}{\partial z}\right) - F_2\frac{\partial}{\partial x}\left(G\frac{\partial H}{\partial z}\right) + F_2\frac{\partial}{\partial z}\left(G\frac{\partial H}{\partial x}\right) - F_3\frac{\partial}{\partial y}\left(G\frac{\partial H}{\partial x}\right)\\ &\quad + F_3\frac{\partial}{\partial x}\left(G\frac{\partial H}{\partial y}\right) - F_1\frac{\partial}{\partial z}\left(G\frac{\partial H}{\partial y}\right) + G\left(\frac{\partial H}{\partial x}\frac{\partial F_3}{\partial y} - \frac{\partial H}{\partial y}\frac{\partial F_3}{\partial x}\right)\\ &\quad + G\left(\frac{\partial H}{\partial y}\frac{\partial F_1}{\partial z} - \frac{\partial H}{\partial z}\frac{\partial F_1}{\partial y}\right) + G\left(\frac{\partial H}{\partial z}\frac{\partial F_2}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial F_2}{\partial z}\right)\\ &= \vec{F}\cdot\left(\vec{\nabla}\times\left(G\vec{\nabla}H\right)\right) + G\,\vec{\nabla}H\cdot\left(\vec{\nabla}\times\vec{F}\right).\end{aligned}$$

According to Formula (3.1) and under the identifications of Paragraph 3.1.1, we then have:

$$\left[\vec{F}, G\,\vec{\nabla}H\right]_S = \vec{F}\cdot\left(\vec{\nabla}G\times\vec{\nabla}H\right) + G\,\vec{\nabla}H\cdot\left(\vec{\nabla}\times\vec{F}\right). \tag{6.16}$$

This permits us to write, with the help of Formula (3.1),

$$\left[F\,\vec{\nabla}L, G\,\vec{\nabla}H\right]_S = F\,\vec{\nabla}L\cdot\left(\vec{\nabla}G\times\vec{\nabla}H\right) + G\,\vec{\nabla}H\cdot\left(\vec{\nabla}F\times\vec{\nabla}L\right). \tag{6.17}$$

This equality and Formula (3.4) imply an identity that will play an important role in our computations: if $F, G, H, L \in \mathcal{A}$, then:

$$\left[F\,\vec{\nabla}L, G\,\vec{\nabla}H\right]_S = -\left[F\,\vec{\nabla}H, G\,\vec{\nabla}L\right]_S. \tag{6.18}$$

According to this equation, we have, for all $l, m \in \mathbf{N}$ and all $1 \le i, j \le \mu - 1$,

$$\left[\varphi^l u_i\vec{\nabla}\varphi, \varphi^m u_j\vec{\nabla}\varphi\right]_S = 0, \qquad \left[\vec{\nabla}u_i, \vec{\nabla}u_j\right]_S = 0, \tag{6.19}$$

while, with the help of (6.17), we obtain,

$$\begin{aligned}\left[\varphi^l u_i\vec{\nabla}\varphi, \vec{\nabla}u_j\right]_S &= \vec{\nabla}u_j\cdot\left(\vec{\nabla}(\varphi^l u_i)\times\vec{\nabla}\varphi\right)\\ &= -\varphi^l\vec{\nabla}\varphi\cdot\left(\vec{\nabla}\times\left(u_i\vec{\nabla}u_j\right)\right)\\ &= \delta^2_\varphi\left(\varphi^l u_i\vec{\nabla}u_j\right).\end{aligned} \tag{6.20}$$

Now, we will use these equations to obtain the formal deformations of the Poisson structure $\{\cdot\,,\cdot\}_\varphi$, where φ is supposed to be weight homogeneous, with an isolated singularity. As we have seen in Proposition 6.10, the family

$$(\varphi^l\, u_i\vec{\nabla}\varphi, \vec{\nabla}u_r,\; l \in \mathbf{N}, i \in \mathcal{E}_\varphi, r = 1, \dots, \mu - 1)$$

gives a $\mathbf{F}$-basis of the Poisson 2-cocycles, modulo the Poisson 2-coboundaries, where the u_j are weight homogeneous polynomials of $\mathcal{A} = \mathbf{F}[x, y, z]$, whose images in $\mathcal{A}_{sing} = \mathbf{F}[x, y, z]/\langle\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\rangle$ give a $\mathbf{F}$-basis of the $\mathcal{A}_{sing}$ $(u_0 = 1)$.

The following proposition gives a formula for all formal deformations of $\{\cdot\,,\cdot\}_\varphi$, up to the equivalence of deformations.

Proposition 6.11. *Let $\varphi \in \mathcal{A} = \mathbf{F}[x, y, z]$ be a weight homogeneous polynomial with an isolated singularity. We consider the Poisson variety $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$ associated to φ, where $\pi_0 = \{\cdot\,,\cdot\}_\varphi$ is the Poisson bracket given by $\{\cdot\,,\cdot\}_\varphi = \frac{\partial\varphi}{\partial x}\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z} + \frac{\partial\varphi}{\partial y}\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial x} + \frac{\partial\varphi}{\partial z}\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}$. This Poisson variety verifies the hypotheses of Proposition 6.8, with:*

$$\begin{aligned}\mathcal{K} &= (\mathbf{N}\times\mathcal{E}_\varphi)\cup\{1, \dots, \mu-1\},\\ \mathbf{a} &= (c^a_{l,i}, \bar{c}^b_r, (l, i) \in \mathbf{N}\times\mathcal{E}_\varphi, 1 \le r \le \mu-1, 1 \le a, b \le n)_{n\in\mathbf{N}^*},\\ \vartheta_{r,j} &= \varphi^r\, u_j\vec{\nabla}\varphi, \quad (r, j) \in \mathbf{N}\times\mathcal{E}_\varphi,\\ \vartheta_i &= \vec{\nabla}u_i, \quad 1 \le i \le \mu-1\\ \Psi^{\bar{\mathbf{a}}_n}_n &= \sum_{\substack{(l,i)\in\mathbf{N}\times\mathcal{E}_\varphi\\ r\in\{1,\dots,\mu-1\}}}\ \sum_{\substack{a+b=n\\ a,b\in\mathbf{N}^*}} c^a_{l,i}\,\bar{c}^b_r\,\varphi^l\, u_i\,\vec{\nabla}u_r.\end{aligned}$$

In particular, for any $c_{l,i}^k \in \mathbf{F}$ and any $\bar{c}_r^k \in \mathbf{F}$ (with $k \in \mathbf{N}^*$, $(l,i) \in \mathbf{N} \times \mathcal{E}_\varphi$ and $1 \le r \le \mu - 1$), the formula

$$\pi_* = \{\cdot\,,\cdot\}_\varphi + \sum_{n\in\mathbf{N}^*} \pi_n \nu^n, \tag{6.21}$$

where for all $n \in \mathbf{N}^*$, π_n is given by:

$$\begin{aligned}\pi_n = &\sum_{\substack{(l,i)\in\mathbf{N}\times\mathcal{E}_\varphi\\ r\in\{1,\ldots,\mu-1\}}} \sum_{\substack{a+b=n\\ a,b\in\mathbf{N}^*}} c_{l,i}^a\, \bar{c}_r^b\, \varphi^l\, u_i\, \vec{\nabla} u_r \\ &+ \sum_{(m,j)\in\mathbf{N}\times\mathcal{E}_\varphi} c_{m,j}^n\, \varphi^m\, u_j \vec{\nabla}\varphi \;+ \sum_{s\in\{1,\ldots,\mu-1\}} \bar{c}_s^n\, \vec{\nabla} u_s,\end{aligned} \tag{6.22}$$

(the sums considered are finite), defines a formal deformation of $\pi_0 = \{\cdot\,,\cdot\}_\varphi$.

Moreover, as a consequence of Proposition 6.8, for any formal deformation π'_* of $\{\cdot\,,\cdot\}_\varphi$, there exist some constants $c_{l,i}^k \in \mathbf{F}$, for $(l,i) \in \mathbf{N} \times \mathcal{E}_\varphi$ and some elements $\bar{c}_r^k \in \mathbf{F}$, for all $1 \le r \le \mu - 1$, $k \in \mathbf{N}^*$, such that π'_* is equivalent to the formal deformation π_* given by the above formulas (6.21) and (6.22).

Proof. According to the fact (See Proposition 6.10) that the elements $\varphi^r\, u_j \vec{\nabla}\varphi$ and $\vec{\nabla} u_i$, $(r,j) \in \mathbf{N} \times \mathcal{E}_\varphi$, $1 \le i \le \mu - 1$ give a $\mathbf{F}$-basis of the second Poisson cohomology space $H^2(\mathcal{A},\varphi)$ of $(\mathcal{A}, \{\cdot\,,\cdot\}_\varphi)$, it suffices to show that equations (6.21) and (6.22) define a formal deformation of $\pi_0 = \{\cdot\,,\cdot\}_\varphi$.

Let us consider some constants $c_{l,i}^k \in \mathbf{F}$, for $(l,i) \in \mathbf{N} \times \mathcal{E}_\varphi$ and some elements $\bar{c}_r^k \in \mathbf{F}$, for all $1 \le r \le \mu - 1$, $k \in \mathbf{N}^*$, and $\pi_* = \{\cdot\,,\cdot\}_\varphi + \sum_{n\in\mathbf{N}^*} \pi_n \nu^n$, with each π_n given by:

$$\begin{aligned}\pi_n = &\sum_{\substack{(l,i)\in\mathbf{N}\times\mathcal{E}_\varphi\\ r\in\{1,\ldots,\mu-1\}}} \sum_{\substack{a+b=n\\ a,b\in\mathbf{N}^*}} c_{l,i}^a\, \bar{c}_r^b\, \varphi^l\, u_i\, \vec{\nabla} u_r \\ &+ \sum_{(m,j)\in\mathbf{N}\times\mathcal{E}_\varphi} c_{m,j}^n\, \varphi^m\, u_j \vec{\nabla}\varphi \;+ \sum_{s\in\{1,\ldots,\mu-1\}} \bar{c}_s^n\, \vec{\nabla} u_s.\end{aligned} \tag{6.23}$$

Let us show that π_* is then a formal deformation of $\pi_0 = \{\cdot\,,\cdot\}_\varphi$. What we have to verify (See equation (6.4)) is that the following equation holds, for any $n \in \mathbf{N}$,

$$\delta_\varphi^2(\pi_{n+1}) = \frac{1}{2} \sum_{\substack{i+j=n+1\\ i,j\ge 1}} [\pi_i, \pi_j]_S\,. \tag{6.24}$$

We have seen that, for $n = 0$, it becomes

$$\delta^2_\varphi(\pi_1) = 0.$$

However, according to the hypothesis (6.23), we have

$$\pi_1 = \sum_{(m,j)\in\mathbf{N}\times\mathcal{E}_\varphi} c^1_{m,j}\,\varphi^m\,u_j\vec{\nabla}\varphi \;+ \sum_{s\in\{1,\ldots,\mu-1\}} \bar{c}^1_s\,\vec{\nabla}u_s,$$

hence $\pi_1 \in Z^2(\mathcal{A},\varphi)$ ($\delta^2_\varphi(\pi_1) = 0$, see Formula (3.14)). Now, assume that $n \geq 1$ and let us prove that the skew-symmetric biderivations $\pi_1, \pi_2, \ldots, \pi_{n+1}$, defined as (6.23), satisfy the equation (6.24).

By using (6.19), one obtains that $\frac{1}{2}\sum\limits_{\substack{i+j=n+1\\ i,j\geq 1}} [\pi_i,\pi_j]_S$ will consist of six types of sums, listed here:

$$1/2 \quad \sum c^a_{l,i}\;\bar{c}^b_r\;c^c_{m,j}\;\bar{c}^d_s\;\left[\varphi^l u_i\vec{\nabla}u_r, \varphi^m u_j\vec{\nabla}u_s\right]_S, \tag{6.25}$$

$$1/2 \quad \sum c^a_{l,i}\;\bar{c}^b_r\;c^q_{m,j}\;\left[\varphi^l u_i\vec{\nabla}u_r, \varphi^m u_j\vec{\nabla}\varphi\right]_S, \tag{6.26}$$

$$1/2 \quad \sum c^c_{l,i}\;\bar{c}^d_r\;c^p_{m,j}\;\left[\varphi^l u_i\vec{\nabla}u_r, \varphi^m u_j\vec{\nabla}\varphi\right]_S, \tag{6.27}$$

$$1/2 \quad \sum \bar{c}^q_r\;c^a_{l,i}\;\bar{c}^b_s\;\left[\varphi^l u_i\vec{\nabla}u_r, \vec{\nabla}u_s\right]_S \tag{6.28}$$

$$1/2 \quad \sum c^c_{l,i}\;\bar{c}^d_r\;\bar{c}^p_s\;\left[\varphi^l u_i\vec{\nabla}u_r, \vec{\nabla}u_s\right]_S \tag{6.29}$$

$$1/2 \sum (c^p_{l,i}\;\bar{c}^q_r + c^q_{l,i}\;\bar{c}^p_r)\;\left[\varphi^l u_i\vec{\nabla}\varphi, \vec{\nabla}u_r\right]_S \tag{6.30}$$

where the sums are taken over the $p, q \in \mathbf{N}$, such that $p+q = n+1$ and $p, q \geq 1$, over the $a, b \in \mathbf{N}$ such that $a+b = p$ and $a, b \geq 1$, over the $c, d \in \mathbf{N}$ such that $c+d = q$ and $c, d \geq 1$ and over the $l, m \in \mathbf{N}$, $i, j \in \mathcal{E}_\varphi$ and $r, s \in \{1, \ldots, \mu-1\}$.

One can observe that for all family of indices $(p,q,a,b,c,d,l,i,r,m,j,s)$, satisfying the conditions above, the indices $(p',q',a',b',c',d',l',i',r',m',j',s')$, defined by:

$$\begin{aligned} p' &= b+c, & a' &= c, & i' &= j,\\ q' &= a+d, & b' &= b, & j' &= i,\\ r' &= r, & c' &= a, & l' &= m,\\ s' &= s, & d' &= d, & m' &= l, \end{aligned}$$

satisfy the same conditions, so that, in the first sum (6.25), one find the element

$$c^a_{l,i}\;\bar{c}^b_r\;c^c_{m,j}\;\bar{c}^d_s\;\left[\varphi^l u_i\vec{\nabla}u_r, \varphi^m u_j\vec{\nabla}u_s\right]_S \tag{6.31}$$

and the element

$$c^{a'}_{l',i'}\;\bar{c}^{b'}_{r'}\;c^{c'}_{m',j'}\;\bar{c}^{d'}_{s'}\;\left[\varphi^{m'} u_{j'}\vec{\nabla}u_{r'}, \varphi^{l'} u_{i'}\vec{\nabla}u_{s'}\right]_S.$$

By definition of the primed indices, one can see that this second element is equal to $c_{l,i}^{a}\ \bar{c}_r^{b}\ c_{m,j}^{c}\ \bar{c}_s^{d}\ \left[\varphi^m u_j\vec{\nabla}u_r, \varphi^l u_i\vec{\nabla}u_s\right]_S$, so that its sum with (6.31) gives zero, according to (6.18). This fact implies that the first sum (6.25) is equal to zero.

With analogous arguments, we will find that all the three sums (6.26), (6.27), (6.28), (6.29) are also equal to zero. Let us give some details about this.

First, for all family of indices $(p, q, a, b, l, i, m, j, r)$, satisfying the conditions required, the family $(p', q', a', b', l', i', m', j', r')$, given by:

$$p' = b + q,\ \ a' = q,\ \ i' = j,\ \ l' = m,$$
$$r' = r,\ \ b' = b,\ \ q' = a,\ \ j' = i,\ \ m' = l,$$

defines another family of indices, with the same conditions that above, so that the second sum (6.26) contains the two elements:

$$c_{l,i}^{a}\ \bar{c}_r^{b}\ c_{m,j}^{q}\ \left[\varphi^l u_i\vec{\nabla}u_r, \varphi^m u_j\vec{\nabla}\varphi\right]_S,$$

and

$$c_{l',i'}^{a'}\ \bar{c}_{r'}^{b'}\ c_{m',j'}^{q'}\ \left[\varphi^{l'} u_{i'}\vec{\nabla}u_{r'}, \varphi^{m'} u_{j'}\vec{\nabla}\varphi\right]_S = c_{m,j}^{q}\ \bar{c}_r^{b}\ c_{l,i}^{a}\ \left[\varphi^m u_j\vec{\nabla}u_r, \varphi^l u_i\vec{\nabla}\varphi\right]_S.$$

As, according to (6.18), this second one is the inverse of the first, the sum (6.26) has to be zero.

The sum (6.27) is of the same type of the sum (6.26), so that it is also equal to zero. Let us consider the sum (6.28), it will be analogous for (6.29), and we will not write the arguments for (6.29). For any family (p, q, a, b, r, s, l, i) with the conditions given above, we can consider another family of indices $(p', q', a', b', r', s', l', i')$, with:

$$p' = a + q,\ \ b' = q,\ \ r' = s,\ \ l' = l,$$
$$a' = a,\ \ q' = b,\ \ s' = r,\ \ i' = i,$$

satifying the same equations, that leads to the fact that, in the sum (6.28), there are the two elements:

$$\bar{c}_r^{q}\ c_{l,i}^{a}\ \bar{c}_s^{b}\ \left[\varphi^l u_i\vec{\nabla}u_r, \vec{\nabla}u_s\right]_S,$$

and

$$\bar{c}_{r'}^{q'}\ c_{l',i'}^{a'}\ \bar{c}_{s'}^{b'}\ \left[\varphi^{l'} u_{i'}\vec{\nabla}u_{r'}, \vec{\nabla}u_{s'}\right]_S = \bar{c}_s^{b}\ c_{l,i}^{a}\ \bar{c}_r^{q}\ \left[\varphi^l u_i\vec{\nabla}u_s, \vec{\nabla}u_r\right]_S,$$

and the sum of these two elements gives zero, because of (6.18), so that the sum (6.28) is zero.

We have then obtained that $\frac{1}{2}\sum\limits_{\substack{i+j=n+1\\ i,j\geq 1}} [\pi_i, \pi_j]_S$ is just given by the sum (6.30), that is to say:

$$\frac{1}{2}\sum_{\substack{i+j=n+1\\ i,j\geq 1}} [\pi_i,\pi_j]_S = 1/2 \sum_{\substack{(l,i)\in\mathbf{N}\times\mathcal{E}_\varphi\\ r\in\{1,\ldots,\mu-1\}}} \sum_{\substack{p+q=n+1\\ p,q\in\mathbf{N}^*}} (c_{l,i}^p\,\bar{c}_r^q + c_{l,i}^q\,\bar{c}_r^p)\left[\varphi^l u_i\vec{\nabla}\varphi, \vec{\nabla}u_r\right]_S$$

$$= 1/2 \sum_{\substack{(l,i)\in\mathbf{N}\times\mathcal{E}_\varphi\\ r\in\{1,\ldots,\mu-1\}}} \sum_{\substack{p+q=n+1\\ p,q\in\mathbf{N}^*}} (c_{l,i}^p\,\bar{c}_r^q + c_{l,i}^q\,\bar{c}_r^p)\ \delta_{\pi_0}^2\left(\varphi^l u_i\vec{\nabla}u_r\right),$$

$$= \sum_{\substack{(l,i)\in\mathbf{N}\times\mathcal{E}_\varphi\\ r\in\{1,\ldots,\mu-1\}}} \sum_{\substack{p+q=n+1\\ p,q\in\mathbf{N}^*}} c_{l,i}^p\,\bar{c}_r^q\ \delta_{\pi_0}^2\left(\varphi^l u_i\vec{\nabla}u_r\right). \tag{6.32}$$

where, for the second equality, we used (6.20). Now, let us consider $\delta_{\pi_0}^2(\pi_{n+1})$. According to the hypothesis (6.23), we have

$$\pi_{n+1} = \sum_{\substack{(l,i)\in\mathbf{N}\times\mathcal{E}_\varphi\\ r\in\{1,\ldots,\mu-1\}}} \sum_{\substack{p+q=n+1\\ p,q\in\mathbf{N}^*}} c_{l,i}^p\,\bar{c}_r^q\,\varphi^l\,u_i\,\vec{\nabla}u_r$$

$$+ \sum_{(m,j)\in\mathbf{N}\times\mathcal{E}_\varphi} c_{m,j}^{n+1}\,\varphi^m\,u_j\vec{\nabla}\varphi\ + \sum_{s\in\{1,\ldots,\mu-1\}} \bar{c}_s^{n+1}\,\vec{\nabla}u_s,$$

so that, according to Proposition 6.10,

$$\pi_{n+1} \in \sum_{\substack{(l,i)\in\mathbf{N}\times\mathcal{E}_\varphi\\ r\in\{1,\ldots,\mu-1\}}} \sum_{\substack{p+q=n+1\\ p,q\in\mathbf{N}^*}} c_{l,i}^p\,\bar{c}_r^q\,\varphi^l\,u_i\,\vec{\nabla}u_r + Z^2(\mathcal{A},\varphi). \tag{6.33}$$

Combining the equations (6.32) and (6.33), we obtain that (6.24) holds, hence the result. □

This Proposition and Remark 6.9 lead to the following interesting result:

Corollary 6.12. *Let $\varphi \in \mathbf{F}[x,y,z]$ be a weight homogeneous polynomial with an isolated singularity. Then, for all $m \in \mathbf{N}^*$, any m-th order deformation of $\{\cdot\,,\cdot\}_\varphi$ extends to a $(m+1)$-order deformation of $\{\cdot\,,\cdot\}_\varphi$.*

We point out that, in general, this property is not satisfied by any Poisson variety, so that the particular family of Poisson varieties associated to weight homogeneous polynomials with an isolated singularity $(\mathbf{F}^3, \{\cdot\,,\cdot\}_\varphi)$ has specific and nice properties of deformations.

6.3 Deformations for surfaces in $\mathbf{F}^3$

In this section, we study the deformations of the Poisson brackets $\{\cdot\,,\cdot\}^\psi$ on $\mathbf{F}^2$ and $\{\cdot\,,\cdot\}_{\mathcal{A}_\varphi}$ on $\mathcal{F}_\varphi$. We will see that, as for $\{\cdot\,,\cdot\}_\varphi$, we can apply Proposition 6.8 in these cases, but in a very easier way.

Deformations of $\{\cdot\,,\cdot\}^\psi$

First, let us deal with the Poisson variety $(\mathbf{F}^2, \{\cdot\,,\cdot\}^\psi)$, equipped with its algebra of regular functions $\mathcal{A} = \mathbf{F}[x,y]$ and where $\psi \in \mathbf{F}[x,y]$ is a square-free weight homogeneous polynomial. We recall that $\{\cdot\,,\cdot\}^\psi$ is the Poisson bracket given by

$$\{\cdot\,,\cdot\}^\psi = \psi\, \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$$

and that we have no skew-symmetric 3-derivations on $\mathbf{F}[x,y]$, $\mathfrak{X}^3(\mathcal{A}) \simeq \{0\}$. So that, if

$$\pi_* = \{\cdot\,,\cdot\}^\psi + \sum_{n\in\mathbf{N}^*} \pi_n \nu^n$$

is a formal deformation of the Poisson bracket $\{\cdot\,,\cdot\}^\psi$, then the equation (See Equation (6.4)), satisfied by π_{m+1}, for any $m \in \mathbf{N}$:

$$\delta^2(\pi_{m+1}) = \frac{1}{2} \sum_{\substack{i+j=m+1\\ i,j\geq 1}} [\pi_i, \pi_j]_S \in \mathfrak{X}^3(\mathcal{A})$$

becomes,in this case:

$$\delta^2(\pi_{m+1}) = 0,$$

where we recall that δ^k denotes the Poisson coboundary operator, associated to $(\mathbf{F}^2, \{\cdot\,,\cdot\}^\psi)$. This fact says, in particular, that any m-th order deformation $\{\cdot\,,\cdot\}^\psi + \pi_1\nu + \cdots + \pi_m\nu^m$ of $\{\cdot\,,\cdot\}^\psi$ ($m \in \mathbf{N}^*$) extends to a $(m+1)$-th order deformation $\{\cdot\,,\cdot\}^\psi + \pi_1\nu + \cdots + \pi_m\nu^m + \pi_{m+1}\nu^{m+1}$ of $\{\cdot\,,\cdot\}^\psi$, by choosing for π_{m+1}, any 2-cocycle of $(\mathbf{F}^2, \{\cdot\,,\cdot\}^\psi)$.

We recall also that we have obtained in Proposition 4.11, a $\mathbf{F}$-basis of the second Poisson cohomology space of $(\mathbf{F}^2, \{\cdot\,,\cdot\}^\psi)$, as we have obtained:

$$H^2(\mathcal{A}, \psi) \simeq \mathcal{A}_{N'(\psi)}\, \psi \;\oplus\; \frac{\mathbf{F}[x,y]}{\left\langle \dfrac{\partial\psi}{\partial x}, \dfrac{\partial\psi}{\partial y} \right\rangle}, \tag{6.34}$$

where $\mathcal{A}_{N'(\psi)}$ is the $\mathbf{F}$-vector space of all the weight homogeneous polynomials of $\mathcal{A} = \mathbf{F}[x,y]$, of degree equal to $N'(\psi) = \varpi(\psi) - \varpi_1 - \varpi_2$ (ϖ_1 and ϖ_2 are the weights of x and y).

Let us denote by μ_ψ the dimension (or Milnor number, see proof of Proposition 4.11) of the $\mathbf{F}$-vector space $\mathcal{A}_{sing}(\psi) = \mathbf{F}[x,y]/\left\langle \dfrac{\partial\psi}{\partial x}, \dfrac{\partial\psi}{\partial y} \right\rangle$ and then by $v_0 = 1, v_1, \ldots, v_{\mu_\psi - 1}$ a family of weight homogeneous polynomials of $\mathbf{F}[x,y]$, whose images in $\mathcal{A}_{sing}(\psi)$ give a $\mathbf{F}$-basis of this $\mathbf{F}$-vector space. Let us also denote by $V_i = x^i y^{N'(\psi)-i}$, for $0 \leq i \leq N'(\psi)$, so that $\{V_0, V_1, \ldots, V_{N'(\psi)}\}$ is a $\mathbf{F}$-basis of $\mathcal{A}_{N'(\psi)}$. Now, we are able to write easily all the formal deformations and m-th order deformations of $\{\cdot\,,\cdot\}^\psi$, up to equivalence, as said in the following:

Proposition 6.13. *Let $\psi \in \mathcal{A} = \mathbf{F}[x,y]$ be a square-free weight homogeneous polynomial. We consider the Poisson variety $(\mathbf{F}^2, \{\cdot\,,\cdot\}^{\psi})$. This Poisson variety verifies the hypotheses of Proposition 6.8, with:*

$$\begin{aligned}
\mathcal{K} &= \{0, \ldots, N'(\psi)\} \times \{0, \ldots, \mu_\psi - 1\},\\
\mathbf{a} &= (\alpha_i^n, \beta_j^n,\ 0 \le i \le N'(\psi), 0 \le j \le \mu_\psi - 1)_{n \in \mathbf{N}^*}\\
\vartheta_i &= V_i = x^i y^{N'(\psi)-i}\,\psi, \quad 0 \le i \le N'(\psi),\\
\vartheta'_j &= v_j, \quad 0 \le j \le \mu_\psi - 1\\
\Psi_n^{\bar{\mathbf{a}}_n} &= 0.
\end{aligned}$$

In particular, for any $\alpha_i^n \in \mathbf{F}$ and any $\beta_j^n \in \mathbf{F}$ (with $n \in \mathbf{N}^*$, $0 \le i \le N'(\psi)$ and $0 \le j \le \mu_\psi - 1$), the formula

$$\pi_* = \{\cdot\,,\cdot\}^{\psi} + \sum_{n \in \mathbf{N}^*} \pi_n \nu^n, \tag{6.35}$$

where for all $n \in \mathbf{N}^*$, π_n is given by:

$$\pi_n = \sum_{i=0}^{N'(\psi)} \alpha_i^n\, V_i\, \psi \ + \sum_{j=0}^{\mu_\psi - 1} \beta_j^n\, v_j, \tag{6.36}$$

defines a formal deformation of $\{\cdot\,,\cdot\}^{\psi}$.

Moreover, as a consequence of Proposition 6.8, for any formal deformation of $\{\cdot\,,\cdot\}^{\psi}$, there exist some constants $\alpha_i^n \in \mathbf{F}$, for $0 \le i \le N'(\psi)$ and some elements $\beta_j^n \in \mathbf{F}$, for all $0 \le j \le \mu_\psi - 1$ ($n \in \mathbf{N}^*$), such that this formal deformation is equivalent to the formal deformation π_* given by the above formulas (6.35) and (6.36).

Deformations of $\{\cdot\,,\cdot\}_{\mathcal{A}_\varphi}$

Finally, we consider the Poisson surface $(\mathcal{F}_\varphi, \{\cdot\,,\cdot\}_{\mathcal{A}_\varphi})$, where $\varphi \in \mathbf{F}[x,y,z]$ is a weight homogeneous polynomial with an isolated singularity and $\mathcal{F}_\varphi$ still denotes the surface $\{\varphi = 0\} \subseteq \mathbf{F}^3$, equipped with its algebra of regular functions $\mathcal{A}_\varphi = \dfrac{\mathbf{F}[x,y,z]}{\langle \varphi \rangle}$.

We have seen in Paragraph 4.3.1, that $\mathfrak{X}^3(\mathcal{A}_\varphi) \simeq \{0\}$. So that, as previously for $(\mathbf{F}^2, \{\cdot\,,\cdot\}^{\psi})$, any m-th order deformation of $\{\cdot\,,\cdot\}_{\mathcal{A}_\varphi}$ extends to a $(m+1)$-th order deformation.

Moreover, in Proposition 4.22, we have obtained that $\{\wp(u_j \vec{\nabla}\varphi), 0 \le j \le \mu - 1 \mid \varpi(u_j) = \varpi(\varphi) - |\varpi|\}$, where μ (or μ_φ) is the Milnor number of φ and $\wp : \mathbf{F}[x,y,z] \to \mathcal{A}_\varphi$ is the natural projection, is a $\mathbf{F}$-basis of the second

Poisson cohomology space of $(\mathcal{F}_\varphi, \{\cdot\,,\cdot\}_{\mathcal{A}_\varphi})$. So that, the same arguments than for the Poisson surface $(\mathbf{F}^2, \{\cdot\,,\cdot\}^\psi)$ lead to the following result, concerning the deformations of $\{\cdot\,,\cdot\}_{\mathcal{A}_\varphi}$:

Proposition 6.14. *Let $\varphi \in \mathbf{F}[x,y,z]$ be a weight homogeneous polynomial with an isolated singularity. We consider the Poisson surface $(\mathcal{F}_\varphi, \{\cdot\,,\cdot\}_{\mathcal{A}_\varphi})$. This Poisson surface verifies the hypotheses of Proposition 6.8, with:*

$$\begin{aligned}
\mathcal{K} &= \{j \in \{0,\ldots,\mu_\varphi - 1\} \mid \varpi(u_j) = \varpi(\varphi) - |\varpi|\},\\
\mathbf{a} &= (\alpha_j^n,\ 0 \le j \le \mu_\varphi - 1 \mid \varpi(u_j) = \varpi(\varphi) - |\varpi|)_{n\in\mathbf{N}^*}\\
\vartheta_j &= \wp(u_j\vec{\nabla}\varphi), \quad 0 \le j \le \mu_\varphi - 1,\ \text{with } \varpi(u_j) = \varpi(\varphi) - |\varpi|,\\
\Psi_n^{\bar{\mathbf{a}}_n} &= 0.
\end{aligned}$$

In particular, for any $\alpha_i^n \in \mathbf{F}$ (with $n \in \mathbf{N}^*$, $0 \le j \le \mu_\varphi - 1$, such that $\varpi(u_j) = \varpi(\varphi) - |\varpi|$), the formula

$$\pi_* = \{\cdot\,,\cdot\}_{\mathcal{A}_\varphi} + \sum_{n\in\mathbf{N}^*} \pi_n \nu^n, \tag{6.37}$$

where for all $n \in \mathbf{N}^*$, π_n is given by:

$$\pi_n = \sum_{\substack{j=0\\ \varpi(u_j)=\varpi(\varphi)-|\varpi|}}^{\mu_\varphi - 1} \alpha_j^n\, \wp(u_j\vec{\nabla}\varphi), \tag{6.38}$$

defines a formal deformation of $\{\cdot\,,\cdot\}_{\mathcal{A}_\varphi}$.

Moreover, as a consequence of Proposition 6.8, for any formal deformation of $\{\cdot\,,\cdot\}_\varphi$, there exist some constants $\alpha_j^n \in \mathbf{F}$, for $0 \le j \le \mu_\varphi - 1$ satisfying $\varpi(u_j) = \varpi(\varphi) - |\varpi|$ ($n \in \mathbf{N}^*$), such that this formal deformation is equivalent to the formal deformation π_* given by the above formulas (6.37) and (6.38).

7

An example of $\chi\{\cdot,\cdot\}_\varphi$

In this section, we consider the affine space $\mathbf{F}^3$, as in Section 3.2, but we want to equip this variety with a Poisson structure more complicated than the Poisson structure $\{\cdot,\cdot\}_\varphi$ (See Paragraph 2.1.3), associated to a polynomial $\varphi \in \mathcal{A} = \mathbf{F}[x,y,z]$. In fact, we would like to add a singularity to the Poisson structure that will not correspond to the singularity of the surface given by the zeros of φ. Our purpose would be to compute the Poisson cohomology of the Poisson algebra $(\mathcal{A}, \chi\{\cdot,\cdot\}_\varphi)$ for any weight homogeneous polynomials $\chi, \varphi \in \mathcal{A}$ where φ would have an isolated singularity and with also a hypothesis on χ. For example, we do not want that $\chi = \varphi^r$, with $r \in \mathbf{N}^*$, otherwise, the Poisson structure $\chi\vec{\nabla}\varphi$ would be $\vec{\nabla}(\varphi^{r+1})$, which is a Poisson structure on the form $\vec{\nabla}\Phi$, that we have studied in Chapter 3, but, $\Phi = \varphi^{r+1}$ is polynomial with a square factor and the methods used in Chapter 3 do not apply in this case.

As we will see, this problem is not easy and we will begin by an example, by considering $\chi = x$ and $\varphi = x^{n+1} + y^{n+1} + z^{n+1}$, that will permit us to show the additional difficulties that appear, in comparison with the case $\chi = 1$ and φ weight homogeneous with an isolated singularity, studied in Section 3.2.

7.1 Poisson complex of $(\mathbf{F}^3, \chi\{\cdot,\cdot\}_\varphi)$

Let us consider the affine space of dimension three $\mathbf{F}^3$, with its algebra of regular functions $\mathcal{A} = \mathbf{F}[x,y,z]$. We have seen in Paragraph 2.1.3, that, to any polynomials $\chi, \varphi \in \mathcal{A}$, we can associate a Poisson structure on $\mathbf{F}^3$, given by

$$\chi\{\cdot,\cdot\}_\varphi = \chi\frac{\partial\varphi}{\partial x}\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z} + \chi\frac{\partial\varphi}{\partial y}\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial x} + \chi\frac{\partial\varphi}{\partial z}\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}.$$

Then, according to the identifications of Paragraph 3.1.1, we can write the Poisson complex associated to the Poisson variety $(\mathbf{F}^3, \chi\{\cdot,\cdot\}_\varphi)$ in terms of elements of $\mathcal{A}$ and $\mathcal{A}^3$, as we have done for the Poisson variety $(\mathbf{F}^3, \{\cdot,\cdot\}_\varphi)$ (See Section 3.2.1). In the case $\chi \neq 1$, we will denote by $\delta^k_{\chi,\varphi}$ the Poisson coboundary operator of the Poisson variety $(\mathbf{F}^3, \chi\{\cdot,\cdot\}_\varphi)$.

Let $F \in \mathcal{A} \simeq \mathfrak{X}^0(\mathcal{A})$ and let us compute the element $\delta^0_{\chi,\varphi}(F) \in \mathfrak{X}^1(\mathcal{A}) \simeq \mathcal{A}^3$.

$$\delta^0_{\chi,\varphi}(F)[x] = \chi\{x, F\}_\varphi = \chi\delta^0_\varphi(F) = \chi\left(\frac{\partial\varphi}{\partial z}\frac{\partial F}{\partial x} - \frac{\partial\varphi}{\partial y}\frac{\partial F}{\partial z}\right),$$

and we have an analogous equality for $\delta^0_{\chi,\varphi}(F)[y]$ and $\delta^0_{\chi,\varphi}(F)[z]$, so that we identify $\delta^0_{\chi,\varphi}(F)$ with the triplet of polynomials

$$\delta^0_{\chi,\varphi}(F) = \chi\,\vec{\nabla}F \times \vec{\nabla}\varphi \in \mathcal{A}^3.$$

We point out that we have, according to the definition of the Poisson coboundary operator (2.12), we have easily, for any $P \in \mathfrak{X}^p(\mathcal{A})$ and any $F_0, F_1, \ldots, F_p \in \mathcal{A}$:

$$\begin{aligned}\delta^p_{\chi,\varphi}(P)[F_0,\ldots,F_p] &= \chi\,\delta^p_\varphi(P)[F_0,\ldots,F_p] \\ &+ \sum_{0\le i<j\le p}(-1)^{i+j}\{F_i,F_j\}_\varphi\, P[\chi, F_0,\ldots,\hat{F_i},\ldots,\hat{F_j},\ldots,F_p].\end{aligned} \tag{7.1}$$

With this formula, it is easy to see that, for $\vec{F} \in \mathcal{A}^3 \simeq \mathfrak{X}^1(\mathcal{A})$, we have

$$\delta^1_{\chi,\varphi}\left(\vec{F}\right) = -\chi\,\vec{\nabla}\left(\vec{F}\cdot\vec{\nabla}\varphi\right) + \chi\ \mathrm{Div}\left(\vec{F}\right)\vec{\nabla}\varphi - \left(\vec{F}\cdot\vec{\nabla}\chi\right)\vec{\nabla}\varphi$$

and, for any $\vec{F} \in \mathcal{A}^3 \simeq \mathfrak{X}^2(\mathcal{A})$,

$$\delta^2_{\chi,\varphi}(\vec{F}) = -\chi\,\vec{\nabla}\varphi\cdot\left(\vec{\nabla}\times\vec{F}\right) - \vec{F}\cdot\left(\vec{\nabla}\chi\times\vec{\nabla}\varphi\right).$$

We deduce from these writings of the Poisson coboundary operator, a writing of the Poisson cohomology spaces, denoted by $H^k(\mathcal{A};\chi,\varphi)$, of the Poisson variety $(\mathbf{F}^3, \chi\{\cdot,\cdot\}_\varphi)$:

$$H^0(\mathcal{A};\chi,\varphi) = \mathrm{Cas}(\mathcal{A};x,\varphi) \simeq \{F \in \mathcal{A} \mid \chi\,\vec{\nabla}F\times\vec{\nabla}\varphi = \vec{0}\},$$

$$H^1(\mathcal{A};\chi,\varphi) \simeq \frac{\{\vec{F}\in\mathcal{A}^3 \mid -\chi\,\vec{\nabla}(\vec{F}\cdot\vec{\nabla}\varphi) + \chi\ \mathrm{Div}(\vec{F})\vec{\nabla}\varphi - \left(\vec{F}\cdot\vec{\nabla}\chi\right)\vec{\nabla}\varphi = \vec{0}\}}{\{\chi\,\vec{\nabla}F\times\vec{\nabla}\varphi \mid F\in\mathcal{A}\}},$$

$$H^2(\mathcal{A};\chi,\varphi) \simeq \frac{\{\vec{F}\in\mathcal{A}^3 \mid \chi\,\vec{\nabla}\varphi\cdot(\vec{\nabla}\times\vec{F}) + \vec{F}\cdot\left(\vec{\nabla}\chi\times\vec{\nabla}\varphi\right) = 0\}}{\{-\chi\,\vec{\nabla}(\vec{F}\cdot\vec{\nabla}\varphi) + \chi\ \mathrm{Div}(\vec{F})\vec{\nabla}\varphi - \left(\vec{F}\cdot\vec{\nabla}\chi\right)\vec{\nabla}\varphi \mid \vec{F}\in\mathcal{A}^3\}},$$

$$H^3(\mathcal{A};\chi,\varphi) \simeq \frac{\mathcal{A}}{\{\chi\,\vec{\nabla}\varphi\cdot(\vec{\nabla}\times\vec{F}) + \vec{F}\cdot\left(\vec{\nabla}\chi\times\vec{\nabla}\varphi\right) \mid \vec{F}\in\mathcal{A}^3\}}.$$

It is clear that, for any $\chi \neq 0$, we have:

$$H^0(\mathcal{A}; \chi, \varphi) \simeq H^0(\mathcal{A}, \varphi) \simeq \bigoplus_{i \in \mathbf{N}} \mathbf{F}\varphi^i,$$

that is to say the Casimirs of $\chi \{\cdot, \cdot\}_\varphi$ are the same as those for $\{\cdot, \cdot\}_\varphi$.

In this chapter, we will determine $H^3(\mathcal{A}; \chi, \varphi)$ and $H^1(\mathcal{A}; \chi, \varphi)$ for the particular case

$$\boxed{\chi = x, \quad \varphi = \varphi_n := \tfrac{1}{n+1}\left(x^{n+1} + y^{n+1} + z^{n+1}\right), \ n \geq 1}.$$

We point out that we are now in the homogeneous context, so that the weights of the three variables are equal to 1 and the Euler derivation is denoted by $\vec{e}$ and given by (under the identifications of paragraph 3.1.1)

$$\vec{e} = (x, y, z) \in \mathcal{A}^3.$$

7.2 The space $H^3(\mathcal{A}; x, \varphi_n)$

Let us study the third Poisson cohomology space $H^3(\mathcal{A}; x, \varphi_n)$, for the Poisson variety $(\mathbf{F}^3, x\{\cdot, \cdot\}_{\varphi_n})$, where φ_n denotes the homogeneous polynomial $\varphi_n = \frac{1}{n+1}\left(x^{n+1} + y^{n+1} + z^{n+1}\right)$, which admits an isolated singularity ($n \geq 1$).

According to the above section 7.1, we have:

$$H^3(\mathcal{A}; x, \varphi_n) \simeq \frac{\mathcal{A}}{\{x\,\vec{\nabla}\varphi_n \cdot (\vec{\nabla} \times \vec{F}) + \vec{F} \cdot \left(\vec{\nabla}x \times \vec{\nabla}\varphi_n\right) \mid \vec{F} \in \mathcal{A}^3\}}.$$

We have, moreover, $\vec{\nabla}x \times \vec{\nabla}\varphi_n = (0, -z^n, y^n) \in \mathcal{A}^3$. Let us begin by given the following result that will appear in the determination of $H^3(\mathcal{A}; x, \varphi_n)$. In this lemma, we will consider the polynomial algebra $\mathbf{F}[y, z]$ and use analogous notations to those introduced in paragraph 4.1.1. Moreover, we denote by $(w_k)_{1 \leq k \leq n^2}$ the family of monomials $y^i z^j$, $0 \leq i \leq n-1$, $0 \leq j \leq n-1$. So that,

$$\frac{\mathbf{F}[y, z]}{\langle y^n, z^n \rangle} \simeq \bigoplus_{k=1}^{n^2} \mathbf{F}w_k \tag{7.2}$$

and we denote by ψ_n the homogeneous polynomial $\psi_n = \frac{1}{n+1}\left(y^{n+1} + z^{n+1}\right)$, so that $\frac{\mathbf{F}[y,z]}{\langle y^n, z^n \rangle}$ is the algebra of regular functions on the surface $\{\psi_n = 0\} \subset \mathbf{F}^2$.

Lemma 7.1. *We have the following isomorphism of* $\mathbf{F}$*-vector spaces:*

$$\frac{\mathbf{F}[y, z]}{\left\{y^n \dfrac{\partial P}{\partial z} - z^n \dfrac{\partial P}{\partial y} \mid P \in \mathbf{F}[y, z]\right\}} \simeq \bigoplus_{i \in \mathbf{N}} \bigoplus_{k=1,\ldots,n^2} \mathbf{F}\,\psi_n^i\, w_k.$$

In other words,

$$\frac{\mathbf{F}[y,z]}{\left\{\vec{\mathcal{H}}_P\cdot\vec{\nabla}\psi_n \mid P\in\mathbf{F}[y,z]\right\}}\simeq\bigoplus_{i\in\mathbf{N}}\bigoplus_{k=1,\ldots,n^2}\mathbf{F}\,\psi_n^i\,w_k,$$

where $\vec{\mathcal{H}}_P=\left(\frac{\partial P}{\partial z},-\frac{\partial P}{\partial y}\right)$ *and* $\vec{\nabla}\psi_n=\left(\frac{\partial\psi_n}{\partial y},\frac{\partial\psi_n}{\partial z}\right)$ *(notations inspired from those in Paragraph 4.1.1).*

Proof. According to (7.2), any homogeneous polynomial $F\in\mathbf{F}[y,z]$ of degree $d\in\mathbf{N}$ can be written as:

$$F=\vec{G}\cdot\vec{\nabla}\psi_n+\sum_{k=1}^{n^2}c_kw_k, \tag{7.3}$$

where $\vec{G}=(G_1,G_2)\in\mathbf{F}[y,z]^2$, with G_1,G_2 two homogeneous polynomials of same degree $d-n$, and where $c_k\in\mathbf{F}$, for $k=1,\ldots,n^2$. Our purpose is to show that there exist $P\in\mathbf{F}[y,z]$ and some constants $\alpha_{j,k}\in\mathbf{F}$, such that

$$F=\vec{\mathcal{H}}_P\cdot\vec{\nabla}\psi_n+\sum_{j\in\mathbf{N}}\sum_{k=1}^{n^2}\alpha_{j,k}\,\psi_n^j\,w_k. \tag{7.4}$$

We will proceed by recursion on the degree $d\in\mathbf{N}$. If $d\leq n$, then $\vec{G}=(a,b)\in\mathbf{F}^2$ and, by writing $\vec{G}=\vec{\mathcal{H}}_{(az-by)}$ in (7.3), we obtain a writing of the form (7.4).

Now, let $d\geq n+1$ and let us suppose that for any homogeneous polynomial $F\in\mathbf{F}[y,z]$ of degree less than $d-1$, an equality of the form (7.4) holds. Let us consider now $F\in\mathbf{F}[y,z]$ a homogeneous polynomial of degree d. We know that F admits a writing of the form (7.3). Then, we have

$$\mathrm{Div}\left(\vec{G}-\frac{1}{d-n+1}\,\mathrm{Div}\left(\vec{G}\right)\vec{e}\right)=0,$$

according to Formula (4.5). Using the exactness of the de Rham complex (See Proposition 4.1), we have the existence of a homogeneous polynomial $Q\in\mathbf{F}[y,z]$, satisfying

$$\vec{G}=\frac{1}{d-n+1}\,\mathrm{Div}\left(\vec{G}\right)\vec{e}+\vec{\mathcal{H}}_Q,$$

so that,

$$\vec{G}\cdot\vec{\nabla}\psi_n=\frac{n+1}{d-n+1}\,\mathrm{Div}\left(\vec{G}\right)\psi_n+\vec{\mathcal{H}}_Q\cdot\vec{\nabla}\psi_n. \tag{7.5}$$

Because the $\mathrm{Div}(\vec{G})\in\mathbf{F}[y,z]$ is a homogeneous polynomial of degree equal to $d-n-1\leq d-1$, by recursion hypothesis, there exist $P\in\mathbf{F}[y,z]$ and some constants $\beta_{i,k}\in\mathbf{F}$, satisfying

$$\text{Div}\left(\vec{G}\right) = \vec{\mathcal{H}}_P \cdot \vec{\nabla}\psi_n + \sum_{i\in\mathbf{N}}\sum_{k=1}^{n^2} \beta_{i,k}\,\psi_n^i\, w_k.$$

Now, by considering Equations (7.3) and (7.5), we obtain:

$$\begin{aligned}
F &= \frac{n+1}{d-n+1}\,\text{Div}\left(\vec{G}\right)\psi_n + \vec{\mathcal{H}}_Q \cdot \vec{\nabla}\psi_n + \sum_{k=1}^{n^2} c_k w_k \\
&= \frac{n+1}{d-n+1}\psi_n\,\vec{\mathcal{H}}_P \cdot \vec{\nabla}\psi_n + \sum_{i\in\mathbf{N}}\sum_{k=1}^{n^2} \frac{n+1}{d-n+1}\beta_{i,k}\,\psi_n^{i+1}\, w_k \\
&\quad + \vec{\mathcal{H}}_Q \cdot \vec{\nabla}\psi_n + \sum_{k=1}^{n^2} c_k w_k \\
&= \vec{\mathcal{H}}_{P'} \cdot \vec{\nabla}\psi_n + \sum_{j\in\mathbf{N}}\sum_{k=1}^{n^2} \alpha_{j,k}\,\psi_n^j\, w_k,
\end{aligned}$$

where $P' = \dfrac{n+1}{d-n+1}\,\psi_n\, P + Q \in \mathbf{F}[y,z]$ and $\alpha_{j,k} = \dfrac{n+1}{d-n+1}\beta_{j-1,k}$, for $j \in \mathbf{N}^*$, while $\alpha_{0,k} = c_k$, hence the result. □

As a consequence of this result, we obtain easily the following analogous in $\mathbf{F}[x,y,z]$.

Proposition 7.2. *Let us still denote by φ_n and ψ_n the homogeneous polynomials $\varphi_n = \frac{1}{n+1}\left(x^{n+1} + y^{n+1} + z^{n+1}\right)$ and $\psi_n = \frac{1}{n+1}\left(y^{n+1} + z^{n+1}\right)$. Then we have*

$$\frac{\mathbf{F}[x,y,z]}{\left\{\left(\vec{\nabla}x \times \vec{\nabla}\varphi_n\right)\cdot\vec{\nabla}P \mid P \in \mathbf{F}[x,y,z]\right\}} \simeq \bigoplus_{\substack{i\in\mathbf{N}\\ j\in\mathbf{N}}} \bigoplus_{k=1,\ldots,n^2} \mathbf{F}\,\psi_n^i\, x^j\, w_k.$$

Proof. Let $F \in \mathbf{F}[x,y,z]$ be a polynomial in the three variables x, y and z. Then, we can write F as follows

$$F = \sum_{i\in\mathbf{N}} x^i F_i(y,z),$$

where, for all $i \in \mathbf{N}$, $F_i \in \mathbf{F}[y,z]$. According to the above Lemma 7.1, there exist some polynomials $P_i \in \mathbf{F}[y,z]$ and some constants $c_{i,j,k} \in \mathbf{F}$ such that, for $i \in \mathbf{N}$,

$$F_i = y^n\frac{\partial P_i}{\partial z} - z^n\frac{\partial P_i}{\partial y} + \sum_{j\in\mathbf{N}}\sum_{k=1,\ldots,n^2} c_{i,j,k}\psi_n^j w_k.$$

As

$$y^n\frac{\partial P_i}{\partial z} - z^n\frac{\partial P_i}{\partial y} = \left(\vec{\nabla}x \times \vec{\nabla}\varphi_n\right)\cdot\vec{\nabla}P_i$$

and

$$x^i\left(\vec{\nabla}x \times \vec{\nabla}\varphi_n\right)\cdot\vec{\nabla}P_i = \left(\vec{\nabla}x \times \vec{\nabla}\varphi_n\right)\cdot\vec{\nabla}\left(x^i P_i\right),$$

we then obtain the desired result. □

Now, let us give a $\mathbf{F}$-basis of the third Poisson cohomology space of $(\mathbf{F}^3, x\{\cdot,\cdot\}_{\varphi_n})$.

Proposition 7.3. *We recall that φ_n denotes the homogeneous polynomial $\varphi_n = \frac{1}{n+1}\left(x^{n+1}+y^{n+1}+z^{n+1}\right)$. The third Poisson cohomology space of the Poisson variety $(\mathbf{F}^3, x\{\cdot,\cdot\}_{\varphi_n})$ is given by:*

$$H^3(\mathcal{A}; x, \varphi_n) \simeq \operatorname{Cas}(\mathcal{A}, \varphi_n) \otimes \frac{\mathbf{F}[x,y,z]}{\langle x^{n+1}, y^n, z^n\rangle},$$

where we recall (from Proposition 3.11) that $\operatorname{Cas}(\mathcal{A}, \varphi_n) = \operatorname{Cas}(\mathcal{A}; x, \varphi_n) = \bigoplus_{i\in\mathbf{N}} \mathbf{F}\varphi_n^i$.

Proof. Let $F \in \mathcal{A} = \mathbf{F}[x,y,z]$ be a homogeneous polynomial. According to Proposition 7.2, there exists $P \in \mathbf{F}[x,y,z]$ such that

$$F \in \left(\vec{\nabla} x \times \vec{\nabla}\varphi_n\right) \cdot \vec{\nabla} P + \sum_{\substack{i\in\mathbf{N}\\ j\in\mathbf{N}}} \sum_{k=1,\ldots,n^2} \mathbf{F}\,\psi_n^i\, x^j\, w_k.$$

We point out that

$$\left(\vec{\nabla} x \times \vec{\nabla}\varphi_n\right) \cdot \vec{\nabla} P = -\delta^2_{x,\varphi_n}(\vec{\nabla} P) \in B^2(\mathcal{A}; x, \varphi_n), \tag{7.6}$$

so that,

$$\mathcal{A} = B^2(\mathcal{A}; x, \varphi_n) + \sum_{\substack{i\in\mathbf{N}\\ j\in\mathbf{N}}} \sum_{k=1,\ldots,n^2} \mathbf{F}\,\psi_n^i\, x^j\, w_k.$$

Now, let us consider the elements $\psi_n^i\, x^j\, w_k$, for $j \geq 2$. For $j \geq 2$, we can write

$$\psi_n^i\, x^j\, w_k = x^2\left(\psi_n^i\, x^{j-2}\, w_k\right)$$

and, according to the determination of $H^3(\mathcal{A}, \varphi_n)$ in Proposition 3.16, there exists $\vec{Q} \in \mathcal{A}^3$, such that

$$\psi_n^i\, x^{j-2}\, w_k \in \vec{\nabla}\varphi_n \cdot \left(\vec{\nabla} \times \vec{Q}\right) + \sum_{\substack{r\in\mathbf{N}\\ 0\leq s\leq \mu-1}} \mathbf{F}\varphi_n^r u_s.$$

As, with the help of Formulas (3.1) and (3.4), $\delta^2_{x,\varphi_n}\left(-x\vec{Q}\right) = x^2\vec{\nabla}\varphi_n \cdot \left(\vec{\nabla} \times \vec{Q}\right)$, we have, for $j \geq 2$,

$$\psi_n^i\, x^j\, w_k \in B^2(\mathcal{A}; x, \varphi_n) + \sum_{\substack{r\in\mathbf{N}\\ 0\leq s\leq \mu-1}} \mathbf{F}\varphi_n^r x^2 u_s.$$

So that, we have now obtain

$$\mathcal{A} = B^2(\mathcal{A}; x, \varphi_n) + \sum_{i\in\mathbf{N}} \sum_{k=1}^{n^2} \mathbf{F}\,\psi_n^i\, w_k + \sum_{j\in\mathbf{N}} \sum_{k=1}^{n^2} \mathbf{F}\,\psi_n^j\, x\, w_k + \sum_{r\in\mathbf{N}} \sum_{s=1}^{\mu-1} \mathbf{F}\,\varphi_n^r x^2 u_s.$$

Let us now study the elements $\psi_n^j\, w_k$ and $\psi_n^j\, x\, w_k$. Our purpose is to relate it to the elements $\varphi_n^j\, w_k$ and $\varphi_n^j\, x\, w_k$.

As $\varphi_n = \psi_n + \frac{1}{n+1}x^{n+1}$, for all $j \in \mathbf{N}$, there exists $\Psi_j \in \mathcal{A}$, satisfying

$$\psi_n^j = \varphi_n^j + x^{n+1}\Psi_j \ \text{ and } \ w_k\, \psi_n^j = w_k\, \varphi_n^j + x^{n+1} w_k\, \Psi_j.$$

Because $n \geq 1$, we can consider $x^{n-1}\Psi_j$ and use one more time the writing of $H^3(\mathcal{A},\varphi_n)$ in Proposition 3.16 to obtain the existence of elements $\vec{R}_{n,j} \in \mathcal{A}^3$, such that

$$x^{n-1} w_k\, \Psi_j \in \vec{\nabla}\varphi_n \cdot \left(\vec{\nabla} \times \vec{R}_{n,j}\right) + \sum_{\substack{r\in\mathbf{N} \\ 0\leq s\leq \mu-1}} \mathbf{F}\varphi_n^r u_s. \tag{7.7}$$

As previously, $x^2\vec{\nabla}\varphi_n \cdot \left(\vec{\nabla} \times \vec{R}_{n,j}\right) = \delta^2_{x,\varphi_n}\left(-x\vec{R}_{n,j}\right) \in B^2(\mathcal{A};x,\varphi_n)$, and we obtain

$$x^{n+1} w_k\, \Psi_j \in B^2(\mathcal{A};x,\varphi_n) + \sum_{\substack{r\in\mathbf{N} \\ 0\leq s\leq \mu-1}} \mathbf{F}\varphi_n^r x^2 u_s$$

and

$$w_k\, \psi_n^j \in B^2(\mathcal{A};x,\varphi_n) + w_k\, \varphi_n^j + \sum_{\substack{r\in\mathbf{N} \\ 0\leq s\leq \mu-1}} \mathbf{F}\varphi_n^r x^2 u_s.$$

Writing $w_k\, x\, \psi_n^j = w_k\, x\, \varphi_n^j + x^{n+2} w_k\, \Psi_j$ and an equality for $x^n w_k\, \Psi_j$, analogous to (7.7), leads to

$$w_k\, x\, \psi_n^j \in B^2(\mathcal{A};x,\varphi_n) + w_k\, x\varphi_n^j + \sum_{\substack{r\in\mathbf{N} \\ 0\leq s\leq \mu-1}} \mathbf{F}\varphi_n^r x^2 u_s.$$

We have then obtained

$$\begin{aligned}
\mathcal{A} &= B^2(\mathcal{A};x,\varphi_n) + \sum_{i\in\mathbf{N}} \sum_{k=1}^{n^2} \mathbf{F}\, \varphi_n^i\, w_k + \sum_{j\in\mathbf{N}} \sum_{k=1}^{n^2} \mathbf{F}\, \varphi_n^j\, x\, w_k \\
&\qquad + \sum_{r\in\mathbf{N}} \sum_{s=1}^{\mu-1} \mathbf{F}\, \varphi_n^r x^2 u_s \\
&= B^2(\mathcal{A};x,\varphi_n) + \sum_{k=1}^{n^2} \mathrm{Cas}(\mathcal{A};x,\varphi_n)\, w_k + \sum_{k=1}^{n^2} \mathrm{Cas}(\mathcal{A};x,\varphi_n)\, x\, w_k \\
&\qquad + \sum_{s=1}^{\mu-1} \mathrm{Cas}(\mathcal{A};x,\varphi_n)\, x^2 u_s,
\end{aligned}$$

where we have seen that $\mathrm{Cas}(\mathcal{A};x,\varphi_n) = \mathrm{Cas}(\mathcal{A},\varphi_n) = \bigoplus_{i\in\mathbf{N}} \mathbf{F}\varphi_n^i$.

We now point out that

$$\{u_s\} = \{w_k, xw_k, \ldots, x^{n-1}w_k\},$$

so that,

$$\{w_k\} \cup \{xw_k\} \cup \{x^2u_s\} = \{w_k, xw_k, \ldots, x^{n-1}w_k, x^n w_k, x^{n+1}w_k\},$$

where $s = 0, \ldots, \mu - 1$, $k = 1, \ldots, n^2$, thus,

$$\mathcal{A} = B^2(\mathcal{A}; x, \varphi_n) + \sum_{k=1}^{n^2} \sum_{i=0}^{n+1} \mathrm{Cas}(\mathcal{A}; x, \varphi_n)\, x^i w_k, \tag{7.8}$$

Finally, let $1 \leq k \leq n^2$ and let $d_k^\circ \in \mathbf{N}$ denote the degree of the monomial w_k. Denoting by $\vec{W}_k \in \mathcal{A}^3$, the element $\vec{W}_k = \left(0, \frac{1}{d_k^\circ+1}\, w_k z, \frac{-1}{d_k^\circ+1}\, w_k y\right)$, we compute

$$\begin{aligned}
\delta^2_{x,\varphi_n}\left(\vec{W}_k\right) &= -x\vec{\nabla}\varphi_n \cdot \left(\vec{\nabla} \times \vec{W}_k\right) - \vec{W}_k \cdot \left(\vec{\nabla}x \times \vec{\nabla}\varphi_n\right) \\
&= -\frac{1}{d_k^\circ + 1} x^{n+1} \left(-2w_k - y\frac{\partial w_k}{\partial y} - z\frac{\partial w_k}{\partial z}\right) \\
&\quad -\frac{1}{d_k^\circ + 1}\left(-z^{n+1}w_k - y^{n+1}w_k\right) \\
&= x^{n+1}w_k + \frac{n+1}{d_k^\circ + 1}\varphi_n w_k,
\end{aligned}$$

where we have used Euler's Formula in $\mathbf{F}[y, z]$, for w_k: $y\frac{\partial w_k}{\partial y} + z\frac{\partial w_k}{\partial z} = d_k^\circ\, w_k$. According to (7.8), that leads to

$$\mathcal{A} = B^2(\mathcal{A}; x, \varphi_n) + \sum_{k=1}^{n^2} \sum_{i=0}^{n} \mathrm{Cas}(\mathcal{A}; x, \varphi_n)\, x^i w_k. \tag{7.9}$$

According to the definition of the elements w_k,

$$\frac{\mathbf{F}[x, y, z]}{\langle x^{n+1}, y^n, z^n\rangle} \simeq \mathbf{F}w_k \oplus \mathbf{F}xw_k \oplus \mathbf{F}x^2 w_k \oplus \cdots \oplus \mathbf{F}x^n w_k, \tag{7.10}$$

so that, it remains to show that the sum in (7.9) is a direct one. To do this, let us suppose the contrary. Then, we denote by $j_0 \in \mathbf{N}$, the smallest integer such that there exist an equation of the form:

$$\sum_{j \geq j_0} \sum_{i=0}^{n} \sum_{k=1}^{n^2} \alpha_{i,j,k}\, x^i \varphi_n^j w_k = \delta^2_{x,\varphi_n}\left(\vec{F}\right) = -x\vec{\nabla}\varphi_n \cdot \left(\vec{\nabla} \times \vec{F}\right) - \vec{F} \cdot \left(\vec{\nabla}x \times \vec{\nabla}\varphi_n\right), \tag{7.11}$$

where $\vec{F} \in \mathcal{A}^3$, $\alpha_{i,j,k} \in \mathbf{F}$ and $\alpha_{i_0,j_0,k_0} \neq 0$, for at least one $0 \leq i_0 \leq n$ and one $1 \leq k_0 \leq n^2$. We will see that such a hypothesis leads to a contradiction.

First, assume that $j_0 = 0$, then Equation (7.11) leads to

$$\begin{aligned}\sum_{i=0}^{n}\sum_{k=1}^{n^2} \alpha_{i,0,k}\, x^i w_k = & \sum_{j\geq 1}\sum_{i=0}^{n}\sum_{k=1}^{n^2} \alpha_{i,j,k}\, x^i \varphi_n^j w_k \\ & -x\vec{\nabla}\varphi_n \cdot \left(\vec{\nabla}\times\vec{F}\right) - \vec{F}\cdot\left(\vec{\nabla}x\times\vec{\nabla}\varphi_n\right) \\ & \in \langle x^{n+1}, y^n, z^n\rangle.\end{aligned}$$

According to (7.10), this implies $\alpha_{i,0,k} = 0$, for all $0 \leq i \leq n$ and all $1 \leq k \leq n^2$, which is in contradiction with the hypothesis.

Now, suppose that $j_0 \geq 1$. Then, the equation

$$\begin{aligned}\sum_{j\geq j_0}\sum_{i=0}^{n}\sum_{k=1}^{n^2} \alpha_{i,j,k}\, x^i \varphi_n^j w_k &= -x\vec{\nabla}\varphi_n \cdot \left(\vec{\nabla}\times\vec{F}\right) - \vec{F}\cdot\left(\vec{\nabla}x\times\vec{\nabla}\varphi_n\right) \\ &= \left(-x\left(\vec{\nabla}\times\vec{F}\right) - \vec{F}\times\vec{\nabla}x\right)\cdot\vec{\nabla}\varphi_n,\end{aligned} \tag{7.12}$$

with the Euler Formula (3.5) and the exactness of the Koszul complex in Proposition 3.5, leads to the existence of an element $\vec{H} \in \mathcal{A}^3$, such that

$$\sum_{j\geq j_0}\sum_{i=0}^{n}\sum_{k=1}^{n^2} \frac{\alpha_{i,j,k}}{n+1}\, x^i \varphi_n^{j-1} w_k \vec{e} = -x\,\vec{\nabla}\times\vec{F} - \vec{F}\times\vec{\nabla}x + \vec{H}\times\vec{\nabla}\varphi_n, \tag{7.13}$$

where, moreover, we can divide $\vec{H}$ by x and obtain a writing as follows

$$\vec{H} = x\,\vec{G} + \vec{L}, \qquad \vec{G}\in\mathcal{A}^3,\ \vec{L}\in\mathbf{F}[y,z]^3. \tag{7.14}$$

Let us compute the inner product of the equation (7.13) with $\vec{\nabla}x$,

$$\begin{aligned}\sum_{j\geq j_0}\sum_{i=0}^{n}\sum_{k=1}^{n^2} \frac{\alpha_{i,j,k}}{n+1}\, x^{i+1} \varphi_n^{j-1} w_k &= -x\,\left(\vec{\nabla}\times\vec{F}\right)\cdot\vec{\nabla}x + \vec{H}\cdot\left(\vec{\nabla}\varphi_n\times\vec{\nabla}x\right) \\ &= -x\,\left(\vec{\nabla}\times\vec{F}\right)\cdot\vec{\nabla}x + x\,\vec{G}\cdot\left(\vec{\nabla}\varphi_n\times\vec{\nabla}x\right) \\ &\quad + \vec{L}\cdot\left(\vec{\nabla}\varphi_n\times\vec{\nabla}x\right).\end{aligned} \tag{7.15}$$

This last equality permits us to write

$$\vec{L}\cdot\left(\vec{\nabla}\varphi_n\times\vec{\nabla}x\right)\in\langle x\rangle,$$

while, as $\vec{L}\in\mathbf{F}[y,z]^3$, we have $\vec{L}\cdot\left(\vec{\nabla}\varphi_n\times\vec{\nabla}x\right)\in\mathbf{F}[y,z]$, so that,

$$\vec{L}\cdot\left(\vec{\nabla}\varphi_n\times\vec{\nabla}x\right) = 0 \tag{7.16}$$

and, dividing by x in (7.15),

$$\sum_{j\geq j_0}\sum_{i=0}^{n}\sum_{k=1}^{n^2}\frac{\alpha_{i,j,k}}{n+1}\,x^i\varphi_n^{j-1}w_k = -\left(\vec{\nabla}\times\vec{F}\right)\cdot\vec{\nabla}x+\vec{G}\cdot\left(\vec{\nabla}\varphi_n\times\vec{\nabla}x\right). \tag{7.17}$$

Moreover, (7.16) can be written as follows

$$L_2 z^n = L_3 y^n,$$

so that, there exists $L\in\mathcal{A}$, such that

$$L_2 = y^n L, \quad L_3 = z^n L. \tag{7.18}$$

Let us now compute the divergence of the equality (7.13) and obtain, using Formulas (3.6) and (3.2),

$$\begin{aligned}\sum_{j\geq j_0}\sum_{i=0}^{n}\sum_{k=1}^{n^2}\frac{\alpha_{i,j,k}}{n+1}&\left(i+(n+1)(j-1)+d_k^\circ+3\right)x^i\varphi_n^{j-1}w_k\\ &= -2\vec{\nabla}x\cdot\left(\vec{\nabla}\times\vec{F}\right)+\left(\vec{\nabla}\times\vec{H}\right)\cdot\vec{\nabla}\varphi_n.\end{aligned} \tag{7.19}$$

As $\vec{H}=x\vec{G}+\vec{L}$, with Formula (3.1), we have

$$\left(\vec{\nabla}\times\vec{H}\right)\cdot\vec{\nabla}\varphi_n = x\left(\vec{\nabla}\times\vec{G}\right)\cdot\vec{\nabla}\varphi_n+\vec{G}\cdot\left(\vec{\nabla}\varphi_n\times\vec{\nabla}x\right)+\left(\vec{\nabla}\times\vec{L}\right)\cdot\vec{\nabla}\varphi_n$$

and, using (7.18) and the fact that $\vec{L}\in\mathbf{F}[y,z]^3$,

$$\begin{aligned}\left(\vec{\nabla}\times\vec{L}\right)\cdot\vec{\nabla}\varphi_n &= x^n\left(\frac{\partial L_3}{\partial y}-\frac{\partial L_2}{\partial z}\right)+y^n\left(\frac{\partial L_1}{\partial z}\right)+z^n\left(-\frac{\partial L_1}{\partial y}\right)\\ &= x^n\left(z^n\frac{\partial L}{\partial y}-y^n\frac{\partial L}{\partial z}\right)+y^n\left(\frac{\partial L_1}{\partial z}\right)+z^n\left(-\frac{\partial L_1}{\partial y}\right)\\ &= -z^n\left(-x^n\frac{\partial L}{\partial y}+\frac{\partial L_1}{\partial y}\right)+y^n\left(-x^n\frac{\partial L}{\partial z}+\frac{\partial L_1}{\partial z}\right)\\ &= \left(\vec{\nabla}x\times\vec{\nabla}\varphi_n\right)\cdot\vec{\nabla}K = \delta^2_{x,\varphi_n}\left(-\vec{\nabla}K\right),\end{aligned}$$

where $K=-x^nL+L_1\in\mathcal{A}$. Let us consider the sum of the equation (7.17), multiplied by -2, and the equation (7.19), to obtain

$$\begin{aligned}\sum_{j\geq j_0}\sum_{i=0}^{n}\sum_{k=1}^{n^2}\frac{\alpha_{i,j,k}}{n+1}&\left(i+(n+1)(j-1)+d_k^\circ+1\right)x^i\varphi_n^{j-1}w_k\\ &= -2\vec{G}\cdot\left(\vec{\nabla}\varphi_n\times\vec{\nabla}x\right)+\left(\vec{\nabla}\times\vec{H}\right)\cdot\vec{\nabla}\varphi_n\end{aligned}$$

$$= -2\vec{G} \cdot \left(\vec{\nabla}\varphi_n \times \vec{\nabla}x\right) + x\left(\vec{\nabla} \times \vec{G}\right) \cdot \vec{\nabla}\varphi_n$$
$$+ \vec{G} \cdot \left(\vec{\nabla}\varphi_n \times \vec{\nabla}x\right) - \delta^2_{x,\varphi_n}\left(\vec{\nabla}K\right)$$
$$= -\vec{G} \cdot \left(\vec{\nabla}\varphi_n \times \vec{\nabla}x\right) + x\left(\vec{\nabla} \times \vec{G}\right) \cdot \vec{\nabla}\varphi_n - \delta^2_{x,\varphi_n}\left(\vec{\nabla}K\right)$$
$$= \delta^2_{x,\varphi_n}\left(-\vec{G} - \vec{\nabla}K\right).$$

We can now write

$$\sum_{j \geq j_0'} \sum_{i=0}^{n} \sum_{k=1}^{n^2} \frac{\alpha_{i,j+1,k}}{n+1}\left(i + (n+1)j + d_k^\circ + 1\right) x^i \varphi_n^j w_k = \delta^2_{x,\varphi_n}\left(-\vec{G} - \vec{\nabla}K\right),$$

where $j_0' = j_0 - 1$, so that, we have obtained an equation of the form (7.12), but with $j_0' < j_0$, that contradicts the hypothesis. We conclude that the sum in (7.9) is direct. □

7.3 The space $H^1(\mathcal{A}; x, \varphi_n)$

In this section, we will determine the first Poisson cohomology space of the Poisson variety $(\mathbf{F}^3, x\{\cdot,\cdot\}_{\varphi_n})$. We recall that we have

$$H^1(\mathcal{A}; x, \varphi_n) \simeq \frac{\{\vec{F} \in \mathcal{A}^3 \mid -x\,\vec{\nabla}(\vec{F} \cdot \vec{\nabla}\varphi_n) + x\,\mathrm{Div}(\vec{F})\vec{\nabla}\varphi_n - \left(\vec{F} \cdot \vec{\nabla}x\right)\vec{\nabla}\varphi_n = \vec{0}\}}{\{x\,\vec{\nabla}F \times \vec{\nabla}\varphi_n \mid F \in \mathcal{A}\}}.$$

We first need the following result:

Lemma 7.4. *We consider the polynomial algebra* $\mathbf{F}[y,z]$*. We recall that we denote by* ψ_n *the homogeneous polynomial* $\psi_n = \frac{1}{n+1}(y^{n+1} + z^{n+1})$*. Let* $K \in \mathbf{F}[y,z]$ *be a homogeneous polynomial satisfying*

$$y^n \frac{\partial K}{\partial z} - z^n \frac{\partial K}{\partial y} = 0.$$

Then, there exist $c \in \mathbf{F}$ *and* $s \in \mathbf{N}$*, such that:*

$$K = c\left(y^{n+1} + z^{n+1}\right)^s = c(n+1)^s \psi_n^s.$$

Proof. The proof of this lemma is inspired by the proof of Proposition 3.11. It is also a particular case of Lemma 4.9. But we will here give a direct proof of this result, for this case. Let $K \in \mathbf{F}[y,z]$ be a homogeneous polynomial of degree $d \in \mathbf{N}$, with $y^n \frac{\partial K}{\partial z} = z^n \frac{\partial K}{\partial y}$. Then, there exists $L \in \mathbf{F}[y,z]$, such that

$$\frac{\partial K}{\partial y} = y^n L, \qquad \frac{\partial K}{\partial z} = z^n L,$$

and Euler's Formula in $\mathbf{F}[y,z]$, $d\,K = \frac{\partial K}{\partial y} y + \frac{\partial K}{\partial z} z$, leads to:

$$d\,K = \frac{\partial K}{\partial y} y + \frac{\partial K}{\partial z} z = (y^n + z^n) L = (n+1)\psi_n L,$$

so that, $K \in \mathbf{F}$ or L divides K. Let us suppose that we have written K as $K = \psi_n^s H$, where $H \in \mathbf{F}[y,z]$ is a homogeneous polynomial, not divisible by ψ_n. We write

$$0 = y^n \frac{\partial K}{\partial z} - z^n \frac{\partial K}{\partial y} = \psi_n^s \left(y^n \frac{\partial H}{\partial z} - z^n \frac{\partial H}{\partial y} \right),$$

so that $y^n \frac{\partial H}{\partial z} - z^n \frac{\partial H}{\partial y} = 0$ and, according to what we have seen previously, $H \in \mathbf{F}$, because, otherwise, ψ_n divides H, that contradicts the hypothesis. Hence the result. □

Before giving a $\mathbf{F}$-basis of $H^1(\mathcal{A}, x\{\cdot,\cdot\}_{\varphi_n})$, we prove another result that will be useful in the determination of this basis.

Lemma 7.5. *We consider the Poisson variety* $(\mathbf{F}^3, x\{\cdot,\cdot\}_{\varphi_n})$*, equipped with its algebra of regular functions* $\mathcal{A} = \mathbf{F}[x,y,z]$ *and we recall that* $\psi_n = \frac{1}{n+1}(y^{n+1} + z^{n+1})$ *and* $\varphi_n = \frac{1}{n+1}(x^{n+1} + y^{n+1} + z^{n+1})$*. For any* $s \in \mathbf{N}$*, we have*

$$\psi_n^s \left(\vec{\nabla} x \times \vec{\nabla} \varphi_n \right) \in \varphi_n^s \left(\vec{\nabla} x \times \vec{\nabla} \varphi_n \right) + B^1(\mathcal{A}; x, \varphi_n).$$

Proof. We will proceed by induction to prove that, for any $s \in \mathbf{N}$, there exists $F_s \in \mathcal{A}$, such that

$$\psi_n^s \left(\vec{\nabla} x \times \vec{\nabla} \varphi_n \right) = \varphi_n^s \left(\vec{\nabla} x \times \vec{\nabla} \varphi_n \right) + \delta^0_{x,\varphi_n}(F_s). \tag{7.20}$$

If $s = 0$, its clear that it suffices to chose $F_s = 0$. Assume now that $r \in \mathbf{N}^*$ and that for all $0 \le s \le r$, there exists $F_s \in \mathcal{A}$ satisfying the equality (7.20). Let us compute

$$\begin{aligned}
\delta^0_{x,\varphi_n}(\psi_n^r) &= x \vec{\nabla}(\psi_n^r) \times \vec{\nabla} \varphi_n \\
&= r\, \psi_n^{r-1}\, x \left(\vec{\nabla} \psi_n \times \vec{\nabla} \varphi_n \right) \\
&= r\, \psi_n^{r-1}\, x\, (0, z^n x^n, -y^n x^n) \\
&= -r\, \psi_n^{r-1}\, x^{n+1} \left(\vec{\nabla} x \times \vec{\nabla} \varphi_n \right) \\
&= -r(n+1)\, \psi_n^{r-1}\, \varphi_n \left(\vec{\nabla} x \times \vec{\nabla} \varphi_n \right) \\
&\quad + r(n+1)\, \psi_n^{r} \left(\vec{\nabla} x \times \vec{\nabla} \varphi_n \right).
\end{aligned}$$

So that, using the induction hypothesis, we have

$$\begin{aligned}\psi_n^r\left(\vec{\nabla}x\times\vec{\nabla}\varphi_n\right) &= \delta^0_{x,\varphi_n}\left(\frac{1}{r(n+1)}\psi_n^r\right) + \varphi_n\,\psi_n^{r-1}\left(\vec{\nabla}x\times\vec{\nabla}\varphi_n\right)\\ &= \delta^0_{x,\varphi_n}\left(\frac{1}{r(n+1)}\psi_n^r\right) + \varphi_n\,\varphi_n^{r-1}\left(\vec{\nabla}x\times\vec{\nabla}\varphi_n\right)\\ &\qquad +\varphi_n\,\delta^0_{x,\varphi_n}\left(F_{r-1}\right)\\ &= \delta^0_{x,\varphi_n}\left(\frac{1}{r(n+1)}\psi_n^r + \varphi_n F_{r-1}\right) + \varphi_n^r\left(\vec{\nabla}x\times\vec{\nabla}\varphi_n\right),\end{aligned}$$

thus,

$$\psi_n^r\left(\vec{\nabla}x\times\vec{\nabla}\varphi_n\right) = \delta^0_{x,\varphi_n}(F_r) + \varphi_n^r\left(\vec{\nabla}x\times\vec{\nabla}\varphi_n\right),$$

where $F_r = \frac{1}{r(n+1)}\psi_n^r + \varphi_n F_{r-1} \in \mathcal{A}$ and we have obtained the equality (7.20), for all $s \in \mathbf{N}$. $\square$

Now, let us give a basis of the $\mathbf{F}$-vector space $H^1(\mathcal{A}; x, \varphi_n)$.

Proposition 7.6. *Let* $(\mathbf{F}^3, x\,\{\cdot\,,\cdot\}_{\varphi_n})$ *be the affine space of dimension three, equipped with the Poisson bracket* $x\,\{\cdot\,,\cdot\}_{\varphi_n}$ *and with its algebra of regular functions* $\mathcal{A} = \mathbf{F}[x, y, z]$. *The first Poisson cohomology space of the Poisson variety* $(\mathbf{F}^3, x\,\{\cdot\,,\cdot\}_{\varphi_n})$ *is given by:*

$$H^1(\mathcal{A}; x, \varphi_n) \simeq \begin{cases} \operatorname{Cas}(\mathcal{A}, \varphi_n)\left(\vec{\nabla}x\times\vec{\nabla}\varphi_n\right) & \text{if } n \neq 1;\\ \operatorname{Cas}(\mathcal{A}, \varphi_n)\,\vec{e}\oplus \operatorname{Cas}(\mathcal{A}, \varphi_n)\left(\vec{\nabla}x\times\vec{\nabla}\varphi_n\right) & \text{if } n = 1,\end{cases} \tag{7.21}$$

where we recall (from Proposition 3.11) that $\operatorname{Cas}(\mathcal{A}, \varphi_n) = \operatorname{Cas}(\mathcal{A}; x, \varphi_n) = \bigoplus_{i\in\mathbf{N}} \mathbf{F}\varphi_n^i$.

Proof. The proof of this result is inspired by the proof of Proposition 3.14. Let $\vec{F} \in \mathcal{A}^3 \simeq \mathfrak{X}^1(\mathcal{A})$ a non zero homogeneous 1-cocycle, i.e., $\vec{F}$ is a triplet of homogeneous polynomials that satisfies

$$0 = \delta^1_{x,\varphi_n}\left(\vec{F}\right) = -x\,\vec{\nabla}\left(\vec{F}\cdot\vec{\nabla}\varphi_n\right) + x\ \operatorname{Div}\left(\vec{F}\right)\vec{\nabla}\varphi_n - \left(\vec{F}\cdot\vec{\nabla}x\right)\vec{\nabla}\varphi_n. \tag{7.22}$$

By computing the cross product with $\vec{\nabla}\varphi_n$, this equation leads to

$$\vec{\nabla}\left(\vec{F}\cdot\vec{\nabla}\varphi_n\right)\times\vec{\nabla}\varphi_n = \vec{0},$$

i.e., $\vec{F}\cdot\vec{\nabla}\varphi_n$ is a Casimir for the Poisson algebra $(\mathcal{A}, \{\cdot\,,\cdot\}_{\varphi_n})$. Proposition 3.11 then implies that there exist $c \in \mathbf{F}$ and $r \in \mathbf{N}^*$, such that

$$\vec{F}\cdot\vec{\nabla}\varphi_n = c\,\varphi_n^r = \frac{c}{n+1}\,\varphi_n^{r-1}\,\vec{\nabla}\varphi_n\cdot\vec{e}, \tag{7.23}$$

where we have used Euler's Formula (3.5). The exactness of the Koszul complex associated to φ_n (Proposition 3.5) permits us to write

$$\vec{F} = \frac{c}{n+1}\varphi_n^{r-1}\vec{e} + \vec{G} \times \vec{\nabla}\varphi_n, \tag{7.24}$$

where $\vec{G} \in \mathcal{A}^3$ is a homogeneous triplet of polynomials. By replacing $\vec{F}\cdot\vec{\nabla}\varphi_n$ by $c\,\varphi_n^r$ (according to (7.23)) in the 1-cocycle condition (7.22), we get,

$$\begin{aligned} 0 &= -x\,\vec{\nabla}\left(\vec{F}\cdot\vec{\nabla}\varphi_n\right) + x\ \mathrm{Div}\left(\vec{F}\right)\vec{\nabla}\varphi_n - \left(\vec{F}\cdot\vec{\nabla}x\right)\vec{\nabla}\varphi_n \\ &= -c\,x\vec{\nabla}\left(\varphi_n^r\right) + x\ \mathrm{Div}\left(\vec{F}\right)\vec{\nabla}\varphi_n - \left(\vec{F}\cdot\vec{\nabla}x\right)\vec{\nabla}\varphi_n \\ &= -cr\,x\varphi_n^{r-1}\vec{\nabla}\varphi_n + x\ \mathrm{Div}\left(\vec{F}\right)\vec{\nabla}\varphi_n - \left(\vec{F}\cdot\vec{\nabla}x\right)\vec{\nabla}\varphi_n, \end{aligned}$$

so that we obtain, by replacing $\vec{F}$ by Formula obtained in (7.24) and using Formulas (3.2) and (3.3),

$$\begin{aligned} 0 &= -cr\,x\varphi_n^{r-1} + \frac{c}{n+1}((n+1)(r-1)+3)\,x\,\varphi_n^{r-1} \\ &\quad + x\,\vec{\nabla}\varphi_n\cdot\left(\vec{\nabla}\times\vec{G}\right) - \frac{c}{n+1}\varphi_n^{r-1}x - \vec{G}\cdot\left(\vec{\nabla}\varphi_n\times\vec{\nabla}x\right) \\ &= -\frac{c(n-1)}{n+1}\varphi_n^{r-1}x + x\,\vec{\nabla}\varphi_n\cdot\left(\vec{\nabla}\times\vec{G}\right) + \vec{G}\cdot\left(\vec{\nabla}x\times\vec{\nabla}\varphi_n\right). \end{aligned}$$

Another writing of this equality is:

$$\frac{c(n-1)}{n+1}\varphi_n^{r-1}x = x\,\vec{\nabla}\varphi_n\cdot\left(\vec{\nabla}\times\vec{G}\right) + \vec{G}\cdot\left(\vec{\nabla}x\times\vec{\nabla}\varphi_n\right) = \delta^2_{x,\varphi_n}\left(-\vec{G}\right).$$

According to Proposition 7.3, we have necessarily $c(n-1) = 0$ and

$$x\,\vec{\nabla}\varphi_n\cdot\left(\vec{\nabla}\times\vec{G}\right) + \vec{G}\cdot\left(\vec{\nabla}x\times\vec{\nabla}\varphi_n\right) = 0. \tag{7.25}$$

We write $\vec{G}$ as follows:

$$\vec{G} = x\vec{H} + \vec{K}, \quad \vec{H}\in\mathcal{A}^3,\ \vec{K}\in\mathbf{F}[y,z]^3$$

and replace this writing in (7.25), thus, with Formula (3.1),

$$\begin{aligned} 0 &= x\,\vec{\nabla}\varphi_n\cdot\left(\vec{\nabla}x\times\vec{H}\right) + x^2\,\vec{\nabla}\varphi_n\cdot\left(\vec{\nabla}\times\vec{H}\right) + x\,\vec{\nabla}\varphi_n\cdot\left(\vec{\nabla}\times\vec{K}\right) \\ &\quad + x\,\vec{H}\cdot\left(\vec{\nabla}x\times\vec{\nabla}\varphi_n\right) + \vec{K}\cdot\left(\vec{\nabla}x\times\vec{\nabla}\varphi_n\right) \\ &= x^2\,\vec{\nabla}\varphi_n\cdot\left(\vec{\nabla}\times\vec{H}\right) + x\,\vec{\nabla}\varphi_n\cdot\left(\vec{\nabla}\times\vec{K}\right) + \vec{K}\cdot\left(\vec{\nabla}x\times\vec{\nabla}\varphi_n\right). \end{aligned}$$

That implies, as $\vec{K}\in\mathbf{F}[y,z]^3$,

$$\vec{K} \cdot \left(\vec{\nabla} x \times \vec{\nabla} \varphi_n\right) \in \mathbf{F}[y,z] \cap \langle x \rangle = \{0\},$$

i.e., $z^n K_2 = y^n K_3$, that leads to the existence of $L \in \mathbf{F}[y,z]$ such that

$$K_2 = y^n L, \quad K_3 = z^n L.$$

We have also, as $\vec{K} \cdot \left(\vec{\nabla} x \times \vec{\nabla} \varphi_n\right) = 0$ and $\vec{K} \in \mathbf{F}[y,z]^3$,

$$\begin{aligned} x\,\vec{\nabla}\varphi_n \cdot \left(\vec{\nabla} \times \vec{H}\right) &= -\vec{\nabla}\varphi_n \cdot \left(\vec{\nabla} \times \vec{K}\right) \\ &= -x^n \left(\frac{\partial K_3}{\partial y} - \frac{\partial K_2}{\partial z}\right) - y^n \frac{\partial K_1}{\partial z} + z^n \frac{\partial K_1}{\partial y}, \end{aligned} \tag{7.26}$$

so that,

$$y^n \frac{\partial K_1}{\partial z} - z^n \frac{\partial K_1}{\partial y} \in \mathbf{F}[y,z] \cap \langle x \rangle = \{0\}.$$

According to Lemma 7.4, there exist $\beta \in \mathbf{F}$ and $s \in \mathbf{N}$ such that

$$K_1 = \beta\,\psi_n^s, \tag{7.27}$$

so that

$$\vec{K} = (\beta\,\psi_n^s,\, y^n L,\, z^n L) = L\vec{\nabla}\varphi_n + (\beta\,\psi_n^s - x^n L)\,\vec{\nabla} x. \tag{7.28}$$

Moreover, equation (7.26) implies

$$\begin{aligned} \vec{\nabla}\varphi_n \cdot \left(\vec{\nabla} \times \vec{H}\right) &= -x^{n-1} \left(\frac{\partial K_3}{\partial y} - \frac{\partial K_2}{\partial z}\right) = -x^{n-1} \left(z^n \frac{\partial L}{\partial y} - y^n \frac{\partial L}{\partial z}\right) \\ &= \vec{\nabla}\varphi_n \cdot \left(\vec{\nabla} \times \left(x^{n-1} L \vec{\nabla} x\right)\right). \end{aligned}$$

We have so obtained

$$\vec{\nabla}\varphi_n \cdot \left(\vec{\nabla} \times \left(\vec{H} - x^{n-1} L \vec{\nabla} x\right)\right) = 0$$

and, according to Corollary 3.9, there exist $P, Q \in \mathcal{A}$, such that:

$$\vec{H} - x^{n-1} L \vec{\nabla} x = \vec{\nabla} P + Q \vec{\nabla}\varphi_n$$

and, with also (7.28),

$$\begin{aligned} \vec{G} &= x\vec{H} + \vec{K} \\ &= x^n L \vec{\nabla} x + x\vec{\nabla} P + xQ\vec{\nabla}\varphi_n + L\vec{\nabla}\varphi_n + (\beta\,\psi_n^s - x^n L)\,\vec{\nabla} x \\ &= x\vec{\nabla} P + xQ\vec{\nabla}\varphi_n + L\vec{\nabla}\varphi_n + \beta\,\psi_n^s\,\vec{\nabla} x. \end{aligned}$$

Now, using Equation (7.24), we have

$$\begin{aligned}\vec{F} &= \frac{c}{n+1}\varphi_n^{r-1}\vec{e} + x\vec{\nabla}P \times \vec{\nabla}\varphi_n + \beta\,\psi_n^s\left(\vec{\nabla}x \times \vec{\nabla}\varphi_n\right)\\ &= \frac{c}{n+1}\varphi_n^{r-1}\vec{e} + \delta^0_{x,\varphi_n}(P) + \beta\,\psi_n^s\left(\vec{\nabla}x \times \vec{\nabla}\varphi_n\right).\end{aligned}$$

Lemma 7.5 then says that

$$\vec{F} \in \frac{c}{n+1}\varphi_n^{r-1}\vec{e} + \mathbf{F}\,\varphi_n^s\left(\vec{\nabla}x \times \vec{\nabla}\varphi_n\right) + B^1(\mathcal{A}; x, \varphi_n).$$

As we have seen that $c(n-1) = 0$, we have:

$$\begin{aligned}&\vec{F} \in B^1(\mathcal{A}; x, \varphi_n)\\ &\quad + \begin{cases} \operatorname{Cas}(\mathcal{A}, \varphi_n)\left(\vec{\nabla}x \times \vec{\nabla}\varphi_n\right) & \text{if } n \neq 1;\\ \operatorname{Cas}(\mathcal{A}, \varphi_n)\,\vec{e} + \operatorname{Cas}(\mathcal{A}, \varphi_n)\left(\vec{\nabla}x \times \vec{\nabla}\varphi_n\right) & \text{if } n = 1.\end{cases}\end{aligned}$$

We have so obtained

$$H^1(\mathcal{A}; x, \varphi_n) \subset \begin{cases} \operatorname{Cas}(\mathcal{A}, \varphi_n)\left(\vec{\nabla}x \times \vec{\nabla}\varphi_n\right) & \text{if } n \neq 1;\\ \operatorname{Cas}(\mathcal{A}, \varphi_n)\,\vec{e} + \operatorname{Cas}(\mathcal{A}, \varphi_n)\left(\vec{\nabla}x \times \vec{\nabla}\varphi_n\right) & \text{if } n = 1.\end{cases} \tag{7.29}$$

The other inclusion is clear as soon as one has verified that, if $n = 1$, then

$$\delta^1_{x,\varphi_n}(\vec{e}) = 0,$$

but

$$\begin{aligned}\delta^1_{x,\varphi_1}(\vec{e}) &= -x\,\vec{\nabla}\left(\vec{e}\cdot\vec{\nabla}\varphi_1\right) + x\,\operatorname{Div}(\vec{e})\,\vec{\nabla}\varphi_1 - \left(\vec{e}\cdot\vec{\nabla}x\right)\vec{\nabla}\varphi_1\\ &= -2x\,\vec{\nabla}\varphi_1 + 3x\,\vec{\nabla}\varphi_1 - x\,\vec{\nabla}\varphi_1\\ &= 0.\end{aligned}$$

It remains to show that the sum in (7.29) is a direct one. To do this, we consider a homogeneous polynomial $F \in \mathcal{A}$ of degree $d \in \mathbf{N}$, some constants $\alpha, \beta \in \mathbf{F}$, with $\alpha = 0$, if $n \neq 1$, and $r, s \in \mathbf{N}$, such that

$$\alpha\,\varphi_n^r\,\vec{e} + \beta\,\varphi_n^s\left(\vec{\nabla}x \times \vec{\nabla}\varphi_n\right) = \delta^0_{x,\varphi_n}(F) = x\left(\vec{\nabla}F \times \vec{\nabla}\varphi_n\right). \tag{7.30}$$

By computing the inner product of this equation with $\vec{\nabla}\varphi_n$, we obtain

$$\alpha\,(n+1)\varphi_n^{r+1} = 0,$$

so that $\alpha = 0$, even if $n = 1$. We now recall that $\psi_n = \frac{1}{n+1}(y^{n+1} + z^{n+1})$. As $\varphi_n = \psi_n + \frac{1}{n+1}x^{n+1}$, for any $s \in \mathbf{N}$, there exists $\Psi_s \in \mathcal{A}$, satisfying

$$\psi_n^s = \varphi_n^s + x^{n+1}\Psi_s.$$

Then, we have

$$x\left(\vec{\nabla}F \times \vec{\nabla}\varphi_n\right) = \beta\,\varphi_n^s\,\left(\vec{\nabla}x \times \vec{\nabla}\varphi_n\right)$$
$$= \beta\,\psi_n^s\,\left(\vec{\nabla}x \times \vec{\nabla}\varphi_n\right) - \beta\,x^{n+1}\Psi_s\,\left(\vec{\nabla}x \times \vec{\nabla}\varphi_n\right).$$

We get

$$\beta\,\psi_n^s\,\left(\vec{\nabla}x \times \vec{\nabla}\varphi_n\right) \in x\mathcal{A}^3 \cap \mathbf{F}[y,z]^3 = \{0\},$$

so $\beta = 0$ and the sum in (7.29) is direct. Hence the desired writing of $H^1(\mathcal{A}; x, \varphi_n)$.
□

Remark 7.7 (Modular derivation). In this section, we have consider the affine space of dimension three $\mathbf{F}^3$, equipped with a Poisson structure $\chi\,\{\cdot\,,\cdot\}_\varphi$, where $\chi, \varphi \in \mathcal{A} = \mathbf{F}[x,y,z]$ are two (weight) homogeneous polynomials. As there exists a volume form $\lambda = \mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z$ in this context, we can compute the modular derivation of the Poisson variety $(\mathbf{F}^3, \chi\,\{\cdot\,,\cdot\}_\varphi)$. We have, by definition (See Paragraph (2.2.3)):

$$*(\mathrm{Div}(\chi\,\{\cdot\,,\cdot\}_\varphi)) = \mathsf{d} * (\chi\,\{\cdot\,,\cdot\}_\varphi) = \mathsf{d}(\chi\,\mathsf{d}\varphi) = \mathsf{d}\chi \wedge \mathsf{d}\varphi,$$

so that, $\mathrm{Div}(\chi\,\{\cdot\,,\cdot\}_\varphi) = \vec{\nabla}\chi \times \vec{\nabla}\varphi \in \mathfrak{X}^1(\mathcal{A})$, under the identifications of Paragraph 3.1.1. We have seen that, for any $n \in \mathbf{N}$, $\vec{\nabla}x \times \vec{\nabla}\varphi_n$ induces a non trivial Poisson cohomology class in $H^1(\mathcal{A}; x, \varphi_n)$.

Moreover, as φ is supposed to be a weight homogeneous polynomial with an isolated singularity, if $\vec{\nabla}\chi \times \vec{\nabla}\varphi = \vec{0}$, according to Proposition 3.11, $\chi \in \mathbf{F}\varphi^r$, with $r \in \mathbf{N}$ and we explained in the introduction of this chapter that we do not want to chose χ in $\mathbf{F}\varphi^r$. So that, we can not apply Proposition 2.28 in the context of this last chapter, but we do not still know if the Poisson cohomology and homology spaces are isomorphic or not.

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