

Height zeta functions on generalized projective toric varieties.

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Abstract. In this paper we study the analytic properties of height zeta functions associated to generalized projective toric varieties. As an application, we obtain asymptotic expansions of the counting functions of rational points of generalized projective toric varieties provided with a large class of heights.

Mathematics Subject Classifications: 11M41 ,14G10, 14G05.

Key words: Manin's conjecture, heights, rational points, zeta functions, meromorphic continuation, Newton polyhedron.

1 Introduction and detailed description of the problem

Let V be a projective algebraic variety over \mathbb{Q} and U a Zariski open subset of V . We can study the density of rational points of a projective embedding of U by first choosing a suitable line bundle \mathcal{L}^1 that determines a projective embedding $\phi : U(\mathbb{Q}) \hookrightarrow \mathbb{P}^n(\mathbb{Q})$ for some n . There are then several ways to measure the density of $\phi U(\mathbb{Q})$. First, we choose a family of local norms $v = (v_p)_{p \leq \infty}$ whose Euler product $H_v = \prod_{p \leq \infty} v_p$ specifies a height H_v on $\phi U(\mathbb{Q})$ in the evident way. It suffices to set $H_{\mathcal{L},v}(M) := H_v(\phi(M))$ for any point $M \in U(\mathbb{Q})$.

By definition, the density of the rational points of $U(\mathbb{Q})$ with respect to $H_{\mathcal{L},v}$ is the function

$$B \rightarrow N_{\mathcal{L},v}(U, B) := \{M \in U(\mathbb{Q}) \mid H_{\mathcal{L},v}(M) \leq B\}.$$

Manin's conjecture (see [8] for more details) concerns the asymptotic behavior of the density for large B . It asserts the existence of constants $a = a(L)$, $b = b(L)$ and $C = C(U, L, v) > 0$, such that:

$$N_{\mathcal{L},v}(U, B) = C B^a (\log B)^{b-1} (1 + o(1)) \quad (B \rightarrow +\infty) \quad (1)$$

A tauberian theorem derive the asymptotic (1) above from the analytic properties of the height-zeta functions defined by the following:

$$Z_{\mathcal{L},v}(U; s) := \sum_{x \in U(\mathbb{Q})} H_{\mathcal{L},v}(x)^{-s}. \quad (2)$$

For nonsingular toric varieties, a suitable refinement of the conjecture was first proved by Batyrev-Tschinkel [2]. Improvements in the error term were then given by Salberger [24], and a bit later by de la Bretèche [5].

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¹technically, one should have that the class of \mathcal{L} in the Picard group of V is contained in the interior of the cone of effective divisors

Each of these works used a *particular* height function H_∞ . The most important feature is its choice of v_∞ , which was defined as follows:

$$\forall \mathbf{y} = (y_1, \dots, y_{n+1}) \in \mathbb{Q}_\infty^{n+1} (= \mathbb{R}^{n+1}), \quad v_\infty(\mathbf{y}) = \max_i |y_i|.$$

Although this is quite convenient to use for torics, there is no fundamental reason why one should a priori limit the effort to prove the conjecture to this particular height function. In addition, since the existence of a precise asymptotic for one height function need not imply anything equally precise for some other height function, a proof of Manin's conjecture for toric varieties and heights other than H_∞ is not a consequence of the work cited above.

In this paper we will study zeta function and the density of rational points on generalized projective toric (also called binomial) varieties equipped with a large class of heights. Following B. Sturmfels (see §1 and lemma 1.1 of [27]), we define here a **generalized projective toric variety** as follows:

$$V(\mathbf{A}) := \{(x_1 : \dots : x_{n+1}) \in \mathbb{P}^n(\mathbb{Q}) \mid \prod_{\substack{j=1, \dots, n+1 \\ a_{i,j} > 0}} x_j^{a_{i,j}} = \prod_{\substack{j=1, \dots, n+1 \\ a_{i,j} < 0}} x_j^{-a_{i,j}} \forall i = 1, \dots, l\}$$

where \mathbf{A} is an $l \times (n+1)$ matrix with entries in \mathbb{Z} , whose rows $\mathbf{a}_i = (a_{i,1}, \dots, a_{i,n+1})$ each satisfy the property that $\sum_{j=1}^{n+1} a_{i,j} = 0$.

*This stands in contrast to common practise in algebraic geometry (see [9]), where toric varieties are assumed to be normal (this follows from the construction of toric varieties from fans). Normality (and therefore smoothness) imposes strong combinatorial restrictions on the matrix \mathbf{A} (see §2 of [27]) for more details). **We don't assume such conditions here** and the reader should consider this fact when comparing our results with those obtained in earlier works cited above.*

The heights or (generalized heights) we consider here are defined as follows:

Let $P = P(X_1, \dots, X_{n+1})$ be a generalized polynomial (i.e. exponents of monomials can be arbitrary nonnegative real numbers) with positive coefficients, elliptic on $[0, \infty)^{n+1}$ ² and homogeneous of degree $d > 0$. To such P , we associate the family of local norms v_p on \mathbb{Q}_p^{n+1} ($p \leq \infty$) by setting, $v_p(\mathbf{y}) = \max |y_j|_p$ for any $p < \infty$ and $\mathbf{y} = (y_1, \dots, y_{n+1})$ and³

$$v_\infty(\mathbf{y}) = (P(|y_1|, \dots, |y_{n+1}|))^{1/d} \quad \forall \mathbf{y} = (y_1, \dots, y_{n+1}) \in \mathbb{Q}_\infty^{n+1}.$$

A simple exercise involving heights then shows that the height H_P associated⁴ to this family of local norms is given as follows. For any $\mathbf{x} = (x_1 : \dots : x_{n+1}) \in \mathbb{P}^n(\mathbb{Q})$ such that $\gcd(x_i) = 1$:

$$H_P(\mathbf{x}) := (P(|x_1|, \dots, |x_{n+1}|))^{1/d}. \quad (3)$$

Very little appears to be known about the asymptotic density of projective varieties for such heights H_P . A few earlier works are [21], [12] or [28], but these are limited to very special choices of P . In particular, the reader should appreciate the fact that none of the general methods developed to

² P is elliptic on $[0, \infty)^{n+1}$ if its restriction to this set vanishes only at $(0, \dots, 0)$. In the following, the term "elliptic" means "elliptic on $[0, \infty)^{n+1}$ ".

³If P is a quadratic form or is of the form $P = X_1^d + \dots + X_{n+1}^d$, then clearly the triangle inequality is verified. However, in order to preserve height under embedding, the definitions of metric used to build heights (see for example ([22], chap. 2, §2.2) or [21]) do not assume in general that the triangle inequality must hold.

⁴Precisely, this is the height associated to the pair $(\iota^* \mathcal{O}(1), v)$ where $\iota : V(\mathbf{A}) \hookrightarrow \mathbb{P}^n(\mathbb{Q})$ is the canonical embedding, $\mathcal{O}(1)$ is the standard line bundle on $\mathbb{P}^n(\mathbb{Q})$, and $v = (v_p)_{p \leq \infty}$ is the family of local norms. For simplicity this height is denoted H_P .

study the asymptotic density with respect to H_∞ can be expected to apply to any other height H_P . To be convinced of this basic fact, it is enough to look at the simple case where $V = \mathbb{P}^n(\mathbb{Q})$ and $U = \{(x_1, \dots, x_{n+1}) \in \mathbb{P}^n(\mathbb{Q}) \mid x_1 \dots x_{n+1} \neq 0\}$. In this case, an easy calculation shows that

$$N_{H_\infty}(U; t) = 2^n \#\{\mathbf{m} \in \mathbb{N}^{n+1} \mid \max m_i \leq t \text{ and } \gcd(m_i) = 1\} = \frac{2^n t^{n+1}}{\zeta(n+1)} + O(t^n). \quad (*1)$$

On the other hand, if $P = P(X_1, \dots, X_{n+1})$ is a homogeneous elliptic polynomial of degree $d > 0$ (for example $P = X_1^d + \dots + X_{n+1}^d$), we have

$$N_{H_P}(U; t) = 2^n \#\{\mathbf{m} \in \mathbb{N}^{n+1} \mid P(\mathbf{m}) \leq t^d \text{ and } \gcd(m_i) = 1\} \sim \frac{2^n}{\zeta(n+1)} \#\{\mathbf{m} \in \mathbb{N}^{n+1} \mid P(\mathbf{m}) \leq t^d\}. \quad (*2)$$

The asymptotic for the second factor in (*2) is given by

$$\#\{\mathbf{m} \in \mathbb{N}^{n+1} \mid P(\mathbf{m}) \leq t^d\} \sim C(P; n)t^{n+1}, \quad \text{where} \quad C(P; n) = \frac{1}{n+1} \int_{\mathbb{S}^n \cap \mathbb{R}_+^{n+1}} P_d^{-(n+1)/d}(\mathbf{v}) d\sigma(\mathbf{v}),$$

\mathbb{S}^n denotes the unit sphere of \mathbb{R}^n , and $d\sigma$ its Lebesgue measure. This is a classical result of Mahler [17]. It should be evident to the reader that this cannot be determined from the asymptotic for the counting function of (*1).

Our results are formulated in terms of a polyhedron in $[0, \infty)^{n+1}$, which can be associated to $V(\mathbf{A})$ in a natural way, and, in addition, the maximal torus $U(\mathbf{A}) := \{\mathbf{x} \in V(\mathbf{A}) : x_1 \dots x_{n+1} \neq 0\}$.

The first result, Theorem 1 (see §3), establishes meromorphic continuation of the height zeta function $Z_{H_P}(U(\mathbf{A}); s) := \sum_{\mathbf{x} \in U(\mathbf{A})} H_P(\mathbf{x})^{-s}$ beyond its domain of convergence and determines explicitly the

principal part at its largest pole (which is very important for applications).

In the case of projective space $\mathbb{P}^n(\mathbb{Q})$ the height zeta function $Z_{H_P}(U(\mathbf{A}); s)$ reduces to the study of Dirichlet's series of the form $Z(Q; s) = \sum_{\mathbf{m} \in \mathbb{N}^n} Q(m_1, \dots, m_n)^{-s}$ where Q is a suitable polynomial and the analytic properties of this last series are closely related to the nature of the **singularity at infinity of the polynomial** Q (see [19], [17], [7], [25], [16], [11],...).

An important feature of Theorem 1 is the precise description for the top order polar term of $Z_{H_P}(U(\mathbf{A}); s)$, in which one sees very clearly the joint dependance upon the polynomial P that defines the height H_P and the geometry of the variety $V(\mathbf{A})$. In my opinion, this joint dependence upon the nature of singularity of P at infinity and the geometry of the variety deserves to be completed in an even more general setting.

As application, Corollary 1 (see §3), shows that if the diagonal intersects the polyhedron in a compact face, then there exist constants $a = a(\mathbf{A})$, $b = b(\mathbf{A})$ and $C = C(\mathbf{A}, H_P)$ such that

$$N_{H_P}(U(\mathbf{A}); t) := \#\{\mathbf{x} \in U(\mathbf{A}) : H_P(\mathbf{x}) \leq t\} = C t^a \log(t)^{b-1} (1 + O((\log t)^{-1})) \text{ as } t \rightarrow \infty.$$

The constants a and b are also characterized quite simply in terms of this polyhedron. The second part of Corollary 1 refines this conclusion by asserting that if the dimension of this face *equals* the dimension of $V(\mathbf{A})$, then $C > 0$. In this event, we are also able to give an explicit expression for C , the form of which is a reasonable generalization of that given in (*2).

Our second main result uses the fact that we are able to give a very precise description of the polyhedron for the class of hypersurfaces $\{\mathbf{x} \in \mathbb{P}^n(\mathbb{Q}) \mid x_1^{a_1} \dots x_n^{a_n} = x_{n+1}^{|\mathbf{a}|}\}$. The particular case of the singular hypersurface $\{\mathbf{x} \in \mathbb{P}^n(\mathbb{Q}) \mid x_1 \dots x_n = x_{n+1}^n\}$, provided with the height H_∞ , was studied in several papers ([2], [3], [13], [14], [1]). However, besides a result of Swinnerton-Dyer [28] for the singular cubic $X_1 X_2 X_3 = X_4^3$ with a single height function *not* H_∞ , nothing comparable to our Theorem 2 appears to exist in the literature.

All our results above follow from our *fundamental* theorem (i.e Theorem 3 in §3.2) which is the main ingredient of this paper. This allows one to study analytic properties for “mixed zeta functions” $Z(f; P; s)$ (see §3.2.2), and in particular, to determine explicitly the principal part at its largest pole. Such zeta functions combine together in one function the multiplicative features of the variety and the additive nature of the polynomial that defines the height H_P .

2 Notations and preliminaries

2.1 Notations

1. $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$;
2. The expression: $f(\lambda, \mathbf{y}, \mathbf{x}) \ll_{\mathbf{y}} g(\mathbf{x})$ uniformly in $\mathbf{x} \in X$ and $\lambda \in \Lambda$ means there exists $A = A(\mathbf{y}) > 0$, such that, $\forall \mathbf{x} \in X$ and $\forall \lambda \in \Lambda$ $|f(\lambda, \mathbf{y}, \mathbf{x})| \leq Ag(\mathbf{x})$;
3. For any $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, we set $\|\mathbf{x}\| = \|\mathbf{x}\|_2 = \sqrt{x_1^2 + \dots + x_n^2}$ and $|\mathbf{x}| = |x_1| + \dots + |x_n|$. We denote the canonical basis of \mathbb{R}^n by $(\mathbf{e}_1, \dots, \mathbf{e}_n)$. The standard inner product on \mathbb{R}^n is denoted by $\langle \cdot, \cdot \rangle$. We set also $\mathbf{0} = (0, \dots, 0)$ and $\mathbf{1} = (1, \dots, 1)$;
4. We denote a vector in \mathbb{C}^n $\mathbf{s} = (s_1, \dots, s_n)$, and write $\mathbf{s} = \boldsymbol{\sigma} + i\boldsymbol{\tau}$, where $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ and $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n)$ are the real resp. imaginary components of \mathbf{s} (i.e. $\sigma_i = \Re(s_i)$ and $\tau_i = \Im(s_i)$ for all i). We also write $\langle \mathbf{x}, \mathbf{s} \rangle$ for $\sum_i x_i s_i$ if $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{s} \in \mathbb{C}^n$;
5. Given $\boldsymbol{\alpha} \in \mathbb{N}_0^n$, we write $\mathbf{X}^{\boldsymbol{\alpha}}$ for the monomial $X_1^{\alpha_1} \dots X_n^{\alpha_n}$. For an analytic function $h(\mathbf{X}) = \sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}} \mathbf{X}^{\boldsymbol{\alpha}}$, the set $\text{supp}(h) := \{\boldsymbol{\alpha} \mid a_{\boldsymbol{\alpha}} \neq 0\}$ is called the support of h ;
6. $f : \mathbb{N}^n \rightarrow \mathbb{C}$ is said to be multiplicative if for all $\mathbf{m} = (m_1, \dots, m_n)$, $\mathbf{m}' = (m'_1, \dots, m'_n) \in \mathbb{N}^n$ satisfying $\gcd(\text{lcm}(m_i), \text{lcm}(m'_i)) = 1$ we have $f(m_1 m'_1, \dots, m_n m'_n) = f(\mathbf{m}) \cdot f(\mathbf{m}')$;
7. Let F be a meromorphic function on a domain \mathcal{D} of \mathbb{C}^n and let \mathcal{S} be the support of its polar divisor. F is said to be of moderate growth if there exists $a, b > 0$ such that $\forall \delta > 0$, $F(\mathbf{s}) \ll_{\boldsymbol{\sigma}, \delta} 1 + |\boldsymbol{\tau}|^{a|\boldsymbol{\sigma}|+b}$ uniformly in $\mathbf{s} = \boldsymbol{\sigma} + i\boldsymbol{\tau} \in \mathcal{D}$ verifying $d(\mathbf{s}, \mathcal{S}) \geq \delta$.

2.2 Preliminaries from convex analysis

2.2.1 Standard constructions

For the reader’s convenience, some classical notions from convex analysis that will be used throughout the article are assembled here. For more details see for example the book [23].

- Let $A = \{\boldsymbol{\alpha}^1, \dots, \boldsymbol{\alpha}^q\}$ be a finite subset of \mathbb{R}^n . The **convex hull** of A is $\text{conv}(A) := \{\sum_{i=1}^q \lambda_i \boldsymbol{\alpha}^i \mid (\lambda_1, \dots, \lambda_q) \in \mathbb{R}_+^q \text{ and } \sum_{i=1}^q \lambda_i = 1\}$ and its interior is $\text{conv}^*(A) := \{\sum_{i=1}^q \lambda_i \boldsymbol{\alpha}^i \mid (\lambda_1, \dots, \lambda_q) \in \mathbb{R}_+^{*q} \text{ and } \sum_{i=1}^q \lambda_i = 1\}$. The **convex cone** of A is $\text{con}(A) := \{\sum_{i=1}^q \lambda_i \boldsymbol{\alpha}^i \mid (\lambda_1, \dots, \lambda_q) \in \mathbb{R}_+^q\}$ and its (relative) interior is $\text{con}^*(A) := \{\sum_{i=1}^q \lambda_i \boldsymbol{\alpha}^i \mid (\lambda_1, \dots, \lambda_q) \in \mathbb{R}_+^{*q}\}$.
- Let $\Sigma = \{\mathbf{x} \in \mathbb{R}_+^n \mid \langle \boldsymbol{\beta}, \mathbf{x} \rangle \geq 1 \ \forall \boldsymbol{\beta} \in I\}$ where I is a finite (nonempty) subset of $\mathbb{R}_+^n \setminus \{\mathbf{0}\}$. Σ is a convex polyhedron of $\mathbb{R}_+^n \setminus \{\mathbf{0}\}$.

1. Let $\mathbf{a} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$, we define $m(\mathbf{a}) := \inf_{\mathbf{x} \in \Sigma} \langle \mathbf{a}, \mathbf{x} \rangle$ and the face of Σ with polar vector \mathbf{a} (or the first meet locus of \mathbf{a}) as $\mathcal{F}_{\Sigma}(\mathbf{a}) = \{\mathbf{x} \in \Sigma \mid \langle \mathbf{a}, \mathbf{x} \rangle = m(\mathbf{a})\}$;
2. The faces of Σ are the sets $\mathcal{F}_{\Sigma}(\mathbf{a})$ ($\mathbf{a} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$). A facet of Σ is a face of maximal dimension;

3. Let F be a face of Σ . The cone $pol(F) := \{\mathbf{a} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\} \mid F = \mathcal{F}_\Sigma(\mathbf{a})\}$ is called the polar cone associated to F and its elements are called polar vectors of F .
A polar vector $\mathbf{a} \in pol(F)$ is said to be a normalized polar vector of F if $m(\mathbf{a}) = 1$ (i.e. $F := \{\mathbf{x} \in \Sigma \mid \langle \mathbf{a}, \mathbf{x} \rangle = 1\}$). We denote by $Pol_0(F)$ the set of normalized polar vector of F ;
4. We define the *index* of Σ by $\iota(\Sigma) := \min\{|\alpha|; \alpha \in \Sigma\}$. It is clear that $\mathcal{F}_\Sigma(\mathbf{1}) = \{\mathbf{x} \in \Sigma : |\mathbf{x}| = \iota(\Sigma)\}$.

• Let J be a subset of $[0, \infty)^n \setminus \{\mathbf{0}\}$, the set $\mathcal{E}(J) = \text{convex hull}(J + \mathbb{R}_+^n)$ is the Newton polyhedron of J . We denote also by $\mathcal{E}^\infty(J) = \text{convex hull}(J - \mathbb{R}_+^n)$ its Newton polyhedron at infinity.

2.3 Construction of the mixed volume constant $A_0(\mathcal{T}; P)$

2.3.1 The volume constant $A_0(P)$

Let $P(\mathbf{X}) = \sum_{\alpha \in \text{supp}(P)} a_\alpha \mathbf{X}^\alpha$ be a generalized polynomial with positive coefficients that depends upon all the variables X_1, \dots, X_n . We apply the discussion in [25] (see also [26]) to define a “volume constant” for P .

By definition, the Newton polyhedron of P (at infinity) is the set $\mathcal{E}^\infty(P) := (\text{conv}(\text{supp}(P)) - \mathbb{R}_+^n)$. All the definitions introduced in §2.2.1 apply to such a polyhedron once we change the definition of $m(\mathbf{a})$ to equal $\sup_{\mathbf{x} \in \mathcal{E}^\infty(P)} \langle \mathbf{a}, \mathbf{x} \rangle$ for any $\mathbf{a} \neq \mathbf{0} \in \mathbb{R}_+^n$. In particular, the notions of face, facet, (normalized) polar vector, etc. all extend in a straightforward way.

Let G_0 be the smallest face of $\mathcal{E}^\infty(P)$ which meets the diagonal $\Delta = \mathbb{R}_+ \mathbf{1}$. We denote by σ_0 the unique positive real number t that satisfies $t^{-1} \mathbf{1} \in G_0$. Thus, there exists a unique vector subspace \vec{G}_0 of largest codimension ρ_0 such that $G_0 \subset \sigma_0^{-1} \mathbf{1} + \vec{G}_0$. Both ρ_0, σ_0 evidently depend upon P , but it is not necessary to indicate this in the notation. We also set $P_{G_0}(X) = \sum_{\alpha \in G_0} a_\alpha \mathbf{X}^\alpha$.

There exist finitely many facets of $\mathcal{E}^\infty(P)$ that intersect in G_0 . We denote their normalized polar vectors by $\lambda_1, \dots, \lambda_N$.

By a permutation of the coordinates X_i one can suppose that $\oplus_{i=1}^{\rho_0} \mathbb{R} \mathbf{e}_i \oplus \vec{G}_0 = \mathbb{R}^n$, and that $\{\mathbf{e}_{m+1}, \dots, \mathbf{e}_n\}$ is the set of standard basis vectors to which G_0 is parallel (i.e. for which $G_0 = G_0 - \mathbb{R}_+ \mathbf{e}_i$). If G_0 is compact then $m = n$.

Set $\Lambda = \text{Conv}\{\mathbf{0}, \lambda_1, \dots, \lambda_N, \mathbf{e}_{\rho_0+1}, \dots, \mathbf{e}_n\}$. It follows that $\dim \Lambda = n$.

Definition 1. *The volume constant associated to P is:*

$$A_0(P) := n! \text{Vol}(\Lambda) \int_{[1, +\infty[^n} \left(\int_{\mathbb{R}_+^{m-\rho_0}} P_{G_0}^{-\sigma_0}(\mathbf{1}, \mathbf{x}, \mathbf{y}) \, d\mathbf{x} \right) d\mathbf{y}.$$

In ([25], chap 3, th. 1.6) (also see [26]), P. Sargos proved the following important result about the function $Y(P; s) := \int_{[1, +\infty[^n} P(\mathbf{x})^{-s} \, d\mathbf{x}$. This generalized earlier work of Cassou-Noguès [7].

Theorem. (P. Sargos [25] chap. 3, see also [26])

Let P be a generalized polynomial with positive coefficients. Then $Y(P; s)$ converges absolutely in $\{\sigma = \Re s > \sigma_0\}$, and has a meromorphic continuation to \mathbb{C} with largest pole at $s = \sigma_0$ of order ρ_0 . In addition, $Y(P; s) \sim_{s \rightarrow \sigma_0} A_0(P) (s - \sigma_0)^{-\rho_0}$. Thus, $A_0(P) > 0$.

Remark. The previous Theorem implies (see [26]), that $\frac{A_0(P)}{\sigma_0(\rho_0-1)!} t^{\sigma_0} \log^{\rho_0-1} t$ equals the dominant term for the volume of the set $\{\mathbf{x} \in [1, \infty)^n \mid P(\mathbf{x}) \leq t\}$ as $t \rightarrow \infty$.

For general results on volume constant near the origin, see also ([10], §5).

When P is elliptic, we recover Mahler’s result.

Proposition 1 ([17]). *If $P = P(X_1, \dots, X_n)$ is an elliptic polynomial of degree $d \geq 1$. Then, $\sigma_0 = \frac{n}{d}$, $\rho_0 = 1$ and $A_0(P) = \frac{1}{d} \int_{\mathbb{S}^{n-1} \cap \mathbb{R}_+^n} P_d^{-n/d}(\mathbf{v}) d\sigma(\mathbf{v})$, where P_d is the homogeneous part of P of degree d , and $d\sigma$ is induced Lebesgue measure on the unit sphere \mathbb{S}^{n-1} .*

2.3.2 Construction of the volume constant $A_0(I, \mathbf{u}, \mathbf{b})$

Let $I = \{\beta_k\}$ be a finite sequence of distinct elements of $\mathbb{R}_+^r \setminus \{\mathbf{0}\}$, $\mathbf{u} = \{u(\beta)\}_{\beta \in I}$ a set of positive integers, and $\mathbf{b} = (b_1, \dots, b_r) \in \mathbb{R}_+^{*r}$. To the triple $(I; \mathbf{u}; \mathbf{b})$, we associate the generalized polynomial $P_{(I; \mathbf{u}; \mathbf{b})}$ in $q := \sum_{\beta \in I} u(\beta)$ variables as follows:

First, we form a sequence $\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2, \dots$ whose elements are the $\beta_k \in I$ but with each element repeated exactly $u(\beta_k)$ times. The indexing of the $\boldsymbol{\alpha}^i$ is specified by the ordering of the β_j as follows. We set $\boldsymbol{\alpha}^1, \dots, \boldsymbol{\alpha}^{u(\beta_1)} = \beta_1$, $\boldsymbol{\alpha}^{u(\beta_1)+1}, \dots, \boldsymbol{\alpha}^{u(\beta_1)+u(\beta_2)} = \beta_2$, etc.

We next form a $q \times r$ matrix whose row vectors are the $\boldsymbol{\alpha}^i$. Denoting its column vectors by $\boldsymbol{\gamma}^1, \dots, \boldsymbol{\gamma}^r$, we obtain r vectors in \mathbb{R}_+^q , with which we now define the generalized polynomial $P_{(I; \mathbf{u}; \mathbf{b})}(\mathbf{Y}) := \sum_{i=1}^r b_i \mathbf{Y}^{\boldsymbol{\gamma}^i}$ on \mathbb{R}_+^q . **We then denote the volume constant of $P_{(I; \mathbf{u}; \mathbf{b})}$ by $A_0(I; \mathbf{u}; \mathbf{b})$.**

2.3.3 The mixed volume constant $A_0(\mathcal{T}; P)$

We now apply the preceding construction by starting with a pair $\mathcal{T} = (I, \mathbf{u})$ of finite sets and a generalized polynomial $P(\mathbf{X}) = b_1 \mathbf{X}^{\boldsymbol{\gamma}^1} + \dots + b_r \mathbf{X}^{\boldsymbol{\gamma}^r}$ on \mathbb{R}_+^n with positive coefficients. We assume that $I \subset \mathbb{R}_+^n \setminus \{\mathbf{0}\}$. The elements of \mathbf{u} are positive integers that depend upon the elements of I . Thus, $\mathbf{u} = \{u(\boldsymbol{\eta})\}_{\boldsymbol{\eta} \in I}$. We also set $\mathbf{b} = (b_1, \dots, b_r)$.

Given the $r \times n$ matrix $\boldsymbol{\Gamma}$ whose row vectors are $\boldsymbol{\gamma}^1, \dots, \boldsymbol{\gamma}^r$, we associate to \mathcal{T} and P the following ‘‘mixed’’ objects I^*, \mathbf{u}^* . These will play an important role.

1. $I^* = \boldsymbol{\Gamma}(I) = \{\boldsymbol{\Gamma}(\boldsymbol{\eta}); \boldsymbol{\eta} \in I\} \subset \mathbb{R}_+^r \setminus \{\mathbf{0}\}$;
2. $\mathbf{u}^* = (u^*(\boldsymbol{\beta}))_{\boldsymbol{\beta} \in I^*}$ where $u^*(\boldsymbol{\beta}) = \sum_{\{\boldsymbol{\Gamma}(\boldsymbol{\eta})=\boldsymbol{\beta}\}} u(\boldsymbol{\eta})$ for each $\boldsymbol{\beta} \in I^*$.

We define finally the **mixed volume** constant by: $A_0(\mathcal{T}; P) = A_0(I^*; \mathbf{u}^*; \mathbf{b})$, the **volume constant** of the generalized polynomial $P_{(I^*; \mathbf{u}^*; \mathbf{b})}$ on \mathbb{R}_+^r (see §2.3.2)

3 Statements of main results

3.1 Main results about height zeta functions on generalized toric varieties

Let \mathbf{A} a $l \times (n+1)$ matrix with entries in \mathbb{Z} , whose rows $\mathbf{a}_i = (a_{i,1}, \dots, a_{i,n+1})$ each satisfy the property that $\sum_{j=1}^{n+1} a_{i,j} = 0$. We consider the generalized projective toric varieties defined by:

$$V(\mathbf{A}) := \{(x_1 : \dots : x_{n+1}) \in \mathbb{P}^n(\mathbb{Q}) \mid \prod_{\substack{j=1, \dots, n+1 \\ a_{i,j} > 0}} x_j^{a_{i,j}} = \prod_{\substack{j=1, \dots, n+1 \\ a_{i,j} < 0}} x_j^{-a_{i,j}} \forall i = 1, \dots, l\} \quad (4)$$

We assume, without loss of generality, that the rows \mathbf{a}_i ($i = 1, \dots, l$) are linearly independent over \mathbb{Q} . It follows that:

$$\text{rank}(\mathbf{A}) = l \text{ and } \dim V(\mathbf{A}) = n - \text{rank}(\mathbf{A}) = n - l. \quad (5)$$

Denote by $U(\mathbf{A}) := \{(x_1 : \dots : x_{n+1}) \in V(\mathbf{A}) : x_1 \dots x_{n+1} \neq 0\}$ the maximal torus of $V(\mathbf{A})$. Define also

$$\begin{aligned} T(\mathbf{A}) &:= \left\{ \nu \in \mathbb{N}_0^{n+1} \mid \mathbf{A}(\nu) = \mathbf{0} \text{ and } \prod_i \nu_i = 0 \right\} \text{ and } T^*(\mathbf{A}) = T(\mathbf{A}) \setminus \{\mathbf{0}\}; \\ c(\mathbf{A}) &:= \frac{1}{2} \# \{ \epsilon \in \{-1, +1\}^{n+1} \mid \prod_{j=1}^{n+1} \epsilon_j^{a_{i,j}} = 1 \forall i \} \end{aligned} \quad (6)$$

Define $\mathcal{E}(\mathbf{A}) := \mathcal{E}(T^*(\mathbf{A})) = \text{convex hull}(T^*(\mathbf{A}) + \mathbb{R}_+^n)$ the Newton polyhedron of $T^*(\mathbf{A})$ and set $\mathcal{F}_0(\mathbf{A})$ to denote the smallest face of $\mathcal{E}(\mathbf{A})$ that meets the diagonal. We then introduce the following:

1. $\rho(\mathbf{A}) := \#(\mathcal{F}_0(\mathbf{A}) \cap T^*(\mathbf{A})) - \dim(\mathcal{F}_0(\mathbf{A}))$;
2. $\mathcal{E}^0(\mathbf{A}) := \{\mathbf{x} \in \mathbb{R}^n \mid \langle \mathbf{x}, \nu \rangle \geq 1 \forall \nu \in \mathcal{E}(\mathbf{A})\}$ (the dual of $\mathcal{E}(\mathbf{A})$);
3. $\iota(\mathbf{A}) := \min\{|\mathbf{c}| \mid \mathbf{c} \in \mathcal{E}^0(\mathbf{A}) \cap \mathbb{R}_+^n\}$ (the index of $\mathcal{E}(\mathbf{A})$).

Fix now a (generalized) polynomial $P = P(X_1, \dots, X_{n+1})$ with positive coefficients and assume that P is elliptic and homogeneous of degree $d > 0$. Denote by H_P the height of $\mathbb{P}^n(\mathbb{Q})$ associated to P (see (3)). We introduce also the following notations:

1. Writing P as a sum of monomials $P(X) = b_1 X^{\gamma^1} + \dots + b_r X^{\gamma^r}$, we set $\mathbf{b} = (b_1, \dots, b_r) \in \mathbb{R}_+^{*r}$
2. Defining $\alpha^1, \dots, \alpha^{n+1} \in \mathbb{R}_+^n$ to be the row vectors of the matrix that equals the transpose of the matrix with rows $\gamma^1, \dots, \gamma^r$, we set $I^*(\mathbf{A}) = \{\sum_{i=1}^n \beta_i \alpha^i \mid \beta \in \mathcal{F}_0(\mathbf{A}) \cap T^*(\mathbf{A})\}$.

We can now state the first result as follows.

Theorem 1. *If the Newton polyhedron $\mathcal{E}(\mathbf{A})$ has a compact face which meets the diagonal, then the height zeta function $\mathbf{s} \mapsto Z_{H_P}(U(\mathbf{A}); \mathbf{s}) := \sum_{M \in U(\mathbf{A})} H_P^{-\mathbf{s}}(M)$ is holomorphic in the half-plane*

$\{s \in \mathbb{C} \mid \sigma > \iota(\mathbf{A})\}$, and there exists $\eta > 0$ such that $s \mapsto Z_{H_P}(U(\mathbf{A}); \mathbf{s})$ has meromorphic continuation with moderate growth to the half-plane $\{s > \iota(\mathbf{A}) - \eta\}$ with only one possible pole at $s = \iota(\mathbf{A})$ of order at most $\rho(\mathbf{A})$.

If we assume in addition that $\dim(\mathcal{F}_0(\mathbf{A})) = \dim V(\mathbf{A}) = n - l$, then $s = \iota(\mathbf{A})$ is indeed a pole of order $\rho(\mathbf{A})$ and $Z_{H_P}(U(\mathbf{A}); \mathbf{s}) \sim_{s \rightarrow \iota(\mathbf{A})} \frac{C_0(\mathbf{A}; H_P)}{(s - \iota(\mathbf{A}))^{\rho(\mathbf{A})}}$, where

$$C_0(\mathbf{A}; H_P) := c(\mathbf{A}) d^{\rho(\mathbf{A})} A_0(I^*(\mathbf{A}); \mathbf{1}; \mathbf{b}) \prod_p \left[\left(1 - \frac{1}{p}\right)^{\#\mathcal{F}_0(\mathbf{A}) \cap T^*(\mathbf{A})} \left(\sum_{\nu \in T(\mathbf{A})} \frac{1}{p^{\langle \nu, \mathbf{c} \rangle}} \right) \right] > 0,$$

$c(\mathbf{A})$ is defined by (6), $A_0(I^(\mathbf{A}); \mathbf{1}; \mathbf{b})$ is the volume constant associated to the polynomial $P_{(I^*(\mathbf{A}), \mathbf{1}, \mathbf{b})}$ (see §2.3.2), and where \mathbf{c} is any⁵ normalized polar vector of the face $\mathcal{F}_0(\mathbf{A})$.*

By a simple adaptation of a standard tauberian argument of Landau (see for example [12], Prop. 3.1), we deduce from Theorem 1 the following arithmetical consequence:

Corollary 1. *If the Newton polyhedron $\mathcal{E}(\mathbf{A})$ has a compact face which meets the diagonal, then there exists a polynomial Q of degree at most $\rho(\mathbf{A}) - 1$ and $\theta > 0$ such that as $t \rightarrow +\infty$:*

$$N_{H_P}(U(\mathbf{A}); t) := \#\{M \in U(\mathbf{A}) \mid H_P(M) \leq t\} = t^{\iota(\mathbf{A})} Q(\log(t)) + O\left(t^{\iota(\mathbf{A}) - \theta}\right).$$

⁵The constant $C_0(\mathbf{A}; H_P)$ does not depend on this choice.

If we assume in addition that $\dim(\mathcal{F}_0(\mathbf{A})) = \dim V(\mathbf{A}) = n - l$, then $Q \neq 0$, $\deg Q = \rho(\mathbf{A}) - 1$ and:

$$N_{H_P}(U(\mathbf{A}); t) = C(\mathbf{A}; H_P) t^{\iota(\mathbf{A})} (\log t)^{\rho(\mathbf{A})-1} (1 + O((\log t)^{-1})) \quad \text{where } C(\mathbf{A}; H_P) := \frac{C_0(\mathbf{A}; H_P)}{\iota(\mathbf{A}) (\rho(\mathbf{A}) - 1)!},$$

($C_0(\mathbf{A}; H_P)$ is the constant volume defined in Theorem 1 above).

Theorem 1 and its corollary 1 are general results that apply to any generalized projective toric variety. Our second result applies Corollary 1 to a particular class of toric hypersurfaces that correspond to a class of problems from multiplicative number theory.

Let $n \in \mathbb{N}$ ($n \geq 2$) and $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$. Set $q = |\mathbf{a}| = a_1 + \dots + a_n$. Consider the hypersurface: $X_n(\mathbf{a}) = \{\mathbf{x} \in \mathbb{P}^n(\mathbb{Q}) \mid x_1^{a_1} \dots x_n^{a_n} = x_{n+1}^q\}$ with torus: $U_n(\mathbf{a}) = \{\mathbf{x} \in X_n(\mathbf{a}) \mid x_1 \dots x_n \neq 0\}$.

Let $P = P(X_1, \dots, X_{n+1})$ be a generalized polynomial as in Theorem 1. Define:

$$\begin{aligned} L_n(\mathbf{a}) &:= \{\mathbf{r} \in \mathbb{N}_0^n \mid q|\langle \mathbf{a}, \mathbf{r} \rangle \text{ and } r_1 \dots r_n = 0\} \setminus \{\mathbf{0}\}; \\ \mathcal{E}(\mathbf{a}) &:= \mathcal{E}(L_n(\mathbf{a})) = \text{convex hull}(L_n(\mathbf{a}) + \mathbb{R}_+^n) \text{ its Newton polyhedron}; \\ \mathcal{F}_0(\mathbf{a}) &= \text{the smallest face of } \mathcal{E}(\mathbf{a}) \text{ which meets the diagonal } \Delta = \mathbb{R}_+ \mathbf{1}; \\ J_n(\mathbf{a}) &:= L_n(\mathbf{a}) \cap \mathcal{F}_0(\mathbf{a}) \quad \text{and} \quad \rho(\mathbf{a}) = \#(J_n(\mathbf{a})) - n + 1; \\ c(\mathbf{a}) &:= \frac{1}{2} \# \{(\varepsilon_1, \dots, \varepsilon_{n+1}) \in \{-1, +1\}^{n+1} \mid \varepsilon_1^{a_1} \dots \varepsilon_n^{a_n} = \varepsilon_{n+1}^q\}. \end{aligned}$$

Then we have:

Theorem 2. *Let $\mathbf{c} \in \mathbb{R}_+^{*n}$ be a normalized polar vector of the face $\mathcal{F}_0(\mathbf{a})$. There exists a polynomial Q of degree at most $\rho(\mathbf{a}) - 1$ and $\theta > 0$ such that:*

$$N_{H_P}(U_n(\mathbf{a}); t) = \#\{M \in U_n(\mathbf{a}) \mid H_P(M) \leq t\} = t^{|\mathbf{c}|} Q(\log t) + O(t^{1-\theta}).$$

If we assume in addition that $\mathcal{F}_0(\mathbf{a})$ is a facet of the polyhedron $\mathcal{E}(\mathbf{a})$, then $Q \neq 0$, $\deg Q = \rho(\mathbf{a}) - 1$ and

$$N_{H_P}(U_n(\mathbf{a}); t) = C(\mathbf{a}; H_P) t^{|\mathbf{c}|} (\log t)^{\rho(\mathbf{a})-1} \times \left(1 + O((\log t)^{-1})\right)$$

where:

$$C(\mathbf{a}; H_P) := \frac{c(\mathbf{a}) d^{\rho(\mathbf{a})} A_0(\mathcal{T}_{\mathbf{c}}; \tilde{P})}{|\mathbf{c}| \cdot (\rho(\mathbf{a}) - 1)!} \cdot \prod_p \left(\left(1 - \frac{1}{p}\right)^{\rho(\mathbf{a})+n-1} \left(\sum_{\substack{\nu \in \mathbb{N}_0^n \\ \nu_1 \dots \nu_n = 0}} p^{-\langle \mathbf{c}, \nu \rangle} \right) \right) > 0,$$

$A_0(\mathcal{T}_{\mathbf{c}}; \tilde{P}) > 0$ is the mixed volume constant (see §2.3.3) associated to the polynomial $\tilde{P}(X_1, \dots, X_n) := P(X_1, \dots, X_n, \prod_{j=1}^n X_j^{a_j/q})$ and to the pair $\mathcal{T}_{\mathbf{c}} = (J_n(\mathbf{a}), (u(\boldsymbol{\beta})_{\boldsymbol{\beta} \in J_n(\mathbf{a})}))$ with $u(\boldsymbol{\beta}) = 1 \forall \boldsymbol{\beta} \in J_n(\mathbf{a})$.

Remark 1: An interesting question is to determine the precise set of exponents $a_1, \dots, a_n \geq 1$ (for given q and n) such that $\mathcal{F}_0(\mathbf{a})$ is a facet of $\mathcal{E}(\mathbf{a})$. It seems reasonable to believe that the complement of this set is thin in a suitable sense (when q is allowed to be arbitrary).

Remark 2: If $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ satisfies the property that each a_i divides $q = a_1 + \dots + a_n$, then $\mathcal{F}_0(\mathbf{a}) = \text{conv}(\{\frac{q}{a_i} \mathbf{e}_i \mid i = 1, \dots, n\})$. Thus, $\mathcal{F}_0(\mathbf{a})$ is a facet of $\mathcal{E}(\mathbf{a})$ and the more precise second part of Theorem 2 applies.

The analogue of Theorem 2 had been proved for the height H_∞ and the particular surface $X_3(\mathbf{1})$ in several earlier works (see [13], [3], [24], [14]). More recently, the article [1] extended these earlier results to $X_n(\mathbf{1})$ for any $n \geq 3$ (but only used H_∞). Theorem 2 should therefore be understood as a natural generalization of all these earlier results.

Remark 3: To illustrate concretely what the theorem is saying, we write out the details when $d = 2$, $n = 3$, and $P = \sum_i X_i^2$. In this event, to apply the discussion in §2.3.3 to find $A_0(\mathcal{T}, \tilde{P})$, we must compute the volume constant $A_0(P_3)$ (see definition 1 in §2.3.1) for the generalized polynomial $P_3 := \tilde{P}_{(J_3^*, \mathbf{u}_3^*, \mathbf{1})}$. An exercise left to the reader will show the following:

$$P_3 = X_1^6 X_4^4 X_5^4 X_6^2 X_8^2 + X_2^6 X_4^2 X_6^4 X_7^4 X_9^2 + X_3^6 X_5^2 X_7^2 X_8^4 X_9^4 + X_1^2 X_2^2 X_3^2 X_4^2 X_5^2 X_6^2 X_7^2 X_8^2 X_9^2.$$

By comparing our result with the result obtained by La Bretèche [3] for the height H_∞ , we get

$$N_{H_P}(U(\mathbb{Q}); t) \sim 2^{11} A_0(P_3) N_{H_\infty}(U(\mathbb{Q}); t) \text{ as } t \rightarrow \infty.$$

Therefore, the constant volume $A_0(P_3)$ associated to the polynomial P_3 defined above, measures the dependency of the counting function on the height.

3.2 Main results about mixed zeta functions

Theorem 1, Corollary 1 and Theorem 2 are simple consequences of our *fundamental* theorem (see Theorem 3 in §3.2.2 below) which is the main ingredient of this paper.

3.2.1 Functions of finite type: Definition and examples

For any arithmetic function $f : \mathbb{N}^n \rightarrow \mathbb{C}$, we can define, at least formally, the Dirichlet series

$$\mathcal{M}(f; \mathbf{s}) := \sum_{\mathbf{m} \in \mathbb{N}^n} \frac{f(m_1, \dots, m_n)}{m^{s_1} \dots m^{s_n}}.$$

Several works (see for example [1], [4], [15], [20]) indicate that the following property should be satisfied by classes of f that are typically encountered in arithmetic problems.

Definition 2. An arithmetic function $f : \mathbb{N}^n \rightarrow \mathbb{C}$ is said to be of **finite type** if there exists a point $\mathbf{c} \in \mathbb{R}_+^{*n}$ such that $\mathcal{M}(f; \mathbf{s})$ converges absolutely if $\sigma_i = \Re(s_i) > c_i \forall i$ and can be continued as a meromorphic function to a neighborhood of \mathbf{c} , as follows:

There exists a pair $\mathcal{T}_{\mathbf{c}} = (I_{\mathbf{c}}, \mathbf{u})$, where $I_{\mathbf{c}}$ is a non empty subset of $\mathbb{R}_+^n \setminus \{\mathbf{0}\}$ and $\mathbf{u} = (u(\boldsymbol{\beta}))_{\boldsymbol{\beta} \in I_{\mathbf{c}}}$ is a vector of positive integers, such that

$$\mathbf{s} \mapsto H(f; \mathcal{T}_{\mathbf{c}}; \mathbf{s}) := \left(\prod_{\boldsymbol{\beta} \in I_{\mathbf{c}}} (\langle \mathbf{s}, \boldsymbol{\beta} \rangle)^{u(\boldsymbol{\beta})} \right) \mathcal{M}(f; \mathbf{c} + \mathbf{s}) \quad (7)$$

has a holomorphic continuation with moderate growth (see (7) §2.1) to the set $\{\mathbf{s} \in \mathbb{C}^n \mid \sigma_i > -\varepsilon_0 \forall i = 1, \dots, n\}$ for some $\varepsilon_0 > 0$.

We can assume⁶ that $\langle \boldsymbol{\beta}, \mathbf{c} \rangle = 1$ for each $\boldsymbol{\beta} \in I_{\mathbf{c}}$. In this case we call $\mathcal{T}_{\mathbf{c}} = (I_{\mathbf{c}}, \mathbf{u})$ a “**regularizing pair**” of $\mathcal{M}(f; \mathbf{s})$ at \mathbf{c} .

If in addition $H(f; \mathcal{T}_{\mathbf{c}}; \mathbf{0}) \neq 0$, we call the pair $\mathcal{T}_{\mathbf{c}}$ the **polar type** of $\mathcal{M}(f; \cdot)$ at \mathbf{c} .

⁶by replacing each vector $\boldsymbol{\beta}$ by the vector $\frac{1}{\langle \boldsymbol{\beta}, \mathbf{c} \rangle} \boldsymbol{\beta}$

3.2.2 Main result in the case of arithmetic functions of finite type

Let $P = P(X_1, \dots, X_n)$ be a generalized polynomial with positive coefficients and of degree $d > 0$. Let $f : \mathbb{N}^n \mapsto \mathbb{C}$ be an arithmetic function of finite type (see §3.2.1).

Let $\mathbf{c} \in \mathbb{R}_+^{*n}$ and $\mathcal{T}_{\mathbf{c}} = (I_{\mathbf{c}}, \mathbf{u})$ a regularizing pair of $\mathcal{M}(f; \cdot)$ at \mathbf{c} as in Definition 2.

$$\text{Set} \quad Z(f; P; s) := \sum_{(m_1, \dots, m_n) \in \mathbb{N}^n} \frac{f(m_1, \dots, m_n)}{P(m_1, \dots, m_n)^{s/d}}.$$

Our results above follow from the following *fundamental* theorem:

Theorem 3. *If P is elliptic and homogeneous, then $s \mapsto Z(f; P; s)$ is holomorphic in $\{s : \sigma > |\mathbf{c}|\}$ and there exists $\eta > 0$ such that $s \mapsto Z(f; P; s)$ has a meromorphic continuation with moderate growth to $\{\sigma > |\mathbf{c}| - \eta\}$ with at most one pole at $s = |\mathbf{c}|$ of order at most $\rho_0(\mathcal{T}_{\mathbf{c}}) := \sum_{\beta \in I_{\mathbf{c}}} u(\beta) - \text{rank}(I_{\mathbf{c}}) + 1$.*

Assume in addition that the following two properties are satisfied:

1. $\mathbf{1} \in \text{con}^*(I_{\mathbf{c}})$;
2. there exists a function K holomorphic in a tubular neighborhood⁷ of $\mathbf{0}$ such that:
 $H(f; \mathcal{T}_{\mathbf{c}}; \mathbf{s}) = K(\langle \beta, \mathbf{s} \rangle)_{\beta \in I_{\mathbf{c}}}$, where $H(f; \mathcal{T}_{\mathbf{c}}; \mathbf{s})$ is the function defined in (7).

Then,

$$Z(f; P; s) = \frac{C_0(f; P)}{(s - |\mathbf{c}|)^{\rho_0(\mathcal{T}_{\mathbf{c}})}} + O\left(\frac{1}{(s - |\mathbf{c}|)^{\rho_0(\mathcal{T}_{\mathbf{c}}) - 1}}\right) \text{ as } s \rightarrow |\mathbf{c}|,$$

where $C_0(f; P) := H(f; \mathcal{T}_{\mathbf{c}}; \mathbf{0}) d^{\rho_0(\mathcal{T}_{\mathbf{c}})} A_0(\mathcal{T}_{\mathbf{c}}, P)$ and where $A_0(\mathcal{T}_{\mathbf{c}}, P) > 0$ is the mixed volume constant associated to P and $\mathcal{T}_{\mathbf{c}}$ (see §2.3.3).

In particular, $s = |\mathbf{c}|$ is a pole of order $\rho_0(\mathcal{T}_{\mathbf{c}})$ if and only if $H(f; \mathcal{T}_{\mathbf{c}}; \mathbf{0}) \neq 0$.

By a simple adaptation of a standard tauberian argument of Landau (see for example [12], Prop. 3.1)), we deduce from Theorem 3 the following arithmetical consequence:

Corollary 2. *If P is as above and $f \geq 0$. There exist a polynomial Q of degree at most $\rho_0(\mathcal{T}_{\mathbf{c}}) - 1$ and $\theta > 0$ such that: $N(f; P^{1/d}; t) := \sum_{\{\mathbf{m} \in \mathbb{N}^n; P^{1/d}(\mathbf{m}) \leq t\}} f(m_1, \dots, m_n) = t^{|\mathbf{c}|} Q(\log t) + O(t^{|\mathbf{c}| - \theta})$.*

In particular, there exists a nonnegative constant $C(f; P)$ such that

$$N(f; P^{1/d}; t) = C(f; P) t^{|\mathbf{c}|} (\log t)^{\rho_0(\mathcal{T}_{\mathbf{c}}) - 1} + O\left(t^{|\mathbf{c}|} (\log t)^{\rho_0(\mathcal{T}_{\mathbf{c}}) - 2}\right) \text{ as } t \rightarrow \infty,$$

Assume in addition that the properties 1 and 2 of theorem 3 are satisfied. Then,

$$C(f; P) := \frac{C_0(f; P)}{|\mathbf{c}| (\rho_0(\mathcal{T}_{\mathbf{c}}) - 1)!} = \frac{H(f; \mathcal{T}_{\mathbf{c}}; \mathbf{0}) d^{\rho_0(\mathcal{T}_{\mathbf{c}})} A_0(\mathcal{T}_{\mathbf{c}}, P)}{|\mathbf{c}| (\rho_0(\mathcal{T}_{\mathbf{c}}) - 1)!} \geq 0,$$

In particular, $C(f; P) > 0$ if and only if $H(f; \mathcal{T}_{\mathbf{c}}; \mathbf{0}) \neq 0$.

Remark 1. To our knowledge, the only result comparable to Corollary 2 is due to La Bretèche [4] who proved estimates for densities $N(f; \|\cdot\|_{\infty}; t)$ using the max norm $\|\mathbf{x}\|_{\infty} = \max_i |x_i|$. Corollary 2 extends his results to a large class of norms or generalized norms. \diamond

⁷By tubular neighborhood of $\mathbf{0}$, we mean a neighborhood of the form $\{\mathbf{s} \in \mathbb{C}^n \mid |\sigma_i| < \varepsilon \forall i\}$ where $\varepsilon > 0$.

Remark 2. Up to normalization factors, the volume constant $C_0(f; P)$ in Theorem 3 is the product of two terms, one arithmetic the other geometric. The arithmetic factor equals $H(f; \mathcal{T}_{\mathbf{c}}; \mathbf{0})$. In the examples that have been worked out, this is typically an eulerian product and depends only on the arithmetic function f . The second part is the mixed volume constant $A_0(\mathcal{T}_{\mathbf{c}}; P)$ which reflects a joint dependence upon both f and P .

An interesting point to be observed here, is that $A_0(\mathcal{T}_{\mathbf{c}}; P)$ is, by definition, the volume constant of a polynomial constructed explicitly from both f and P (see §2.3.3). This polynomial has in general more variables than the original polynomial P .

Remark 3. Assumption (2) of Theorem 3 is always satisfied if $\text{rank } I_{\mathbf{c}} = n$.

Remark 4. It is clear that any monomial $f(\mathbf{m}) = m_1^{\mu_1} \dots m_n^{\mu_n}$ is of finite type since $\mathcal{M}(f; \mathbf{s}) = \prod_{i=1}^n \zeta(s_i - \mu_i)$. Thus, the polar type of $\mathcal{M}(f; \cdot)$ at $\mathbf{c} = (1 + \mu_1, \dots, 1 + \mu_n)$ is $\mathcal{T}_{\mathbf{c}} = (I; \mathbf{u})$ where $I = \{\frac{1}{\mu_i+1} \mathbf{e}_i : i = 1, \dots, n\}$ and $u(\beta) = 1$ for all $\beta \in I$. It's easy to see that all assumptions of Corollary 2 are satisfied. In particular, for $\boldsymbol{\mu} = \mathbf{0}$ (i.e $f \equiv 1$), Corollary 2 implies that:

$$\#\{\mathbf{m} \in \mathbb{N}^n \mid \mathcal{N}_P(\mathbf{m}) \leq x\} = \frac{1}{n} \left(\int_{\mathbb{S}^{n-1} \cap \mathbb{R}_+^n} P^{-n/d}(\mathbf{u}) \, d\sigma(\mathbf{u}) \right) x^n + O(x^{n-\delta}) \text{ as } x \rightarrow \infty. \quad (8)$$

Therefore conclusion of Corollary 2 agree with that obtained by Mahler [17] (see Proposition 1) and Sargos (see [25] chap 3 or [26]).

3.2.3 Uniform multiplicative functions

The Theorem 3 (and its corollary 2) above applies to a large class of arithmetic functions f . However, to use this theorem we need to choose a suitable singular point \mathbf{c} in the boundary of the domain of convergence of $\mathcal{M}(f; \mathbf{s})$ to determine its polar type $\mathcal{T}_{\mathbf{c}}$ and to verify all the request assumptions. In general, It is not easy to choose such suitable point \mathbf{c} and this question was not trated by La Bretèche in [4]. In proposition 2 below we will adress this problem for the class of uniform multiplicative functions f which are sufficient to prove our Main results about rational points on generalized projective toric varieties .

Definition 3. A multiplicative function $f : \mathbb{N}^n \mapsto \mathbb{N}_0$ is said to be **uniform** if there exists a function $g_f : \mathbb{N}_0^n \mapsto \mathbb{N}_0$ and two constants $M, C > 0$ such that for all prime numbers p and all $\boldsymbol{\nu} \in \mathbb{N}_0^n$, $f(p^{\nu_1}, \dots, p^{\nu_n}) = g_f(\boldsymbol{\nu}) \leq C(1 + |\boldsymbol{\nu}|)^M$.

We fix a uniform multiplicative function $f : \mathbb{N}^n \mapsto \mathbb{N}_0$ throughout the rest of §3.2.3 and write g in place of g_f . We then define:

1. $S^*(g) = \{\boldsymbol{\nu} \in \mathbb{N}_0^n \setminus \{\mathbf{0}\} \mid g(\boldsymbol{\nu}) \neq 0\}$ and assume that $S^*(g) \neq \emptyset$;
2. $\mathcal{E}(f) := \mathcal{E}(S^*(g)) = \text{convex hull}(S^*(g) + \mathbb{R}_+^n)$ the Newton polyhedron determined by $S^*(g)$;
3. $\mathcal{E}(f)^o := \{\mathbf{x} \in \mathbb{R}^n \mid \langle \mathbf{x}, \boldsymbol{\nu} \rangle \geq 1 \, \forall \boldsymbol{\nu} \in S^*(g)\}$ the dual of $\mathcal{E}(f)$;
4. $\iota(f) := \min \{|\mathbf{c}| \mid \mathbf{c} \in \mathcal{E}(f)^o \cap \mathbb{R}_+^n\}$ (the ‘‘index’’ of f);
5. $\mathcal{F}_0(f) :=$ the smallest face of $\mathcal{E}(f)$ which meets the diagonal. We denote its set of normalized polar vectors by $\text{pol}_0(\mathcal{F}_0(f))$;

With theses notations we have:

Proposition 2. *Let $\mathbf{c} \in \text{pol}_0(\mathcal{F}_0(f))$ be a normalized polar vector of the face $\mathcal{F}_0(f)$. Set $\mathcal{T}_{\mathbf{c}} := (I_f; \mathbf{u}_f)$ where $I_f := \mathcal{F}_0(f) \cap S^*(g)$ and $\mathbf{u}_f := (g(\beta))_{\beta \in I_f}$. Then, the multiple zeta function*

$$\mathcal{M}(f; \mathbf{s}) = \sum_{m_1, \dots, m_n \geq 1} \frac{f(m_1, \dots, m_n)}{m_1^{s_1} \dots m_n^{s_n}} \text{ converges absolutely in } \{\mathbf{s} \mid \Re(s_i) > c_i \ \forall i\} \text{ and there exists}$$

$\varepsilon_0 > 0$ such that $\mathbf{s} \mapsto H(f; \mathcal{T}_{\mathbf{c}}; \mathbf{s}) = \left(\prod_{\beta \in \mathcal{F}_0(f) \cap S^*(g)} (\langle \beta, \mathbf{s} \rangle)^{g(\beta)} \right) \mathcal{M}(f; \mathbf{c} + \mathbf{s})$ has a holomorphic continuation with moderate growth to $\{\mathbf{s} \in \mathbb{C}^n \mid \forall i \ \Re(s_i) > -\varepsilon_0\}$ and satisfies

$$H(f; \mathcal{T}_{\mathbf{c}}; \mathbf{0}) = \prod_p \left(1 - \frac{1}{p} \right)^{\sum_{\nu \in \mathcal{F}_0(f) \cap S^*(g)} g(\nu)} \left(\sum_{\nu \in \mathbb{N}_0^n} \frac{g(\nu)}{p^{\langle \nu, \mathbf{c} \rangle}} \right) > 0.$$

In particular f is of finite type, $\mathcal{T}_{\mathbf{c}} = (I_f; \mathbf{u}_f)$ is the polar type of $\mathcal{M}(f; \cdot)$ in \mathbf{c} and $\mathbf{1} \in \text{con}^*(I_f)$.

If we assume in addition that $\dim \mathcal{F}_0(f) = \text{rank}(S^*(g)) - 1$, there exists a function K holomorphic in a tubular neighborhood of $\mathbf{0}$ such that: $H(f; \mathcal{T}_{\mathbf{c}}; \mathbf{s}) = K(\langle \beta, \mathbf{s} \rangle)_{\beta \in I_{\mathbf{c}}}$.

Remark: The assumption $\dim \mathcal{F}_0(f) = \text{rank}(S^*(g)) - 1$ is automatically satisfied if for example the face $\mathcal{F}_0(f)$ is a facet of $\mathcal{E}(f)$.

4 Proof of Theorem 3

The starting point of our method is the remarkable formula of Mellin:

$$\frac{\Gamma(s)}{(\sum_{k=0}^r w_k)^s} = \frac{1}{(2\pi i)^r} \int_{\rho_1 - i\infty}^{\rho_1 + i\infty} \dots \int_{\rho_r - i\infty}^{\rho_r + i\infty} \frac{\Gamma(s - z_1 - \dots - z_r) \prod_{i=1}^r \Gamma(z_i) dz}{w_0^{s - z_1 - \dots - z_r} (\prod_{k=1}^r w_k^{z_k})} \quad (9)$$

valid if $\forall i = 0, \dots, r, \Re(w_i) > 0, \forall i = 1, \dots, r \ \rho_i > 0$ and $\Re(s) > \rho_1 + \dots + \rho_r$.

Methods that use Mellin's formula in the classical case do not adapt easily to prove Theorem 3. The main reason this appears to be the case is that the inductive procedure, in which one inducts on the number of monomials in an expression for P , is incapable of the precision we need to prove our main result, that is, an explicit description of the top order term in the principal part of $Z(f; P; s)$ at its first pole. So, in order to prove theorem 3, our strategy is the following: First, we use Mellin's formula (9) (with a suitable integer r) to write $Z(f; P; s)$ as an integral of a twist of $\mathcal{M}(f; \mathbf{s})$ over a chain of \mathbb{C}^r (see section §4.2 bellow). This is needed because the arithmetical's informations needed to derive informations on the possible first pole σ_0 are contained in the polar divisor of the arithmetical series $\mathcal{M}(f; \mathbf{s})$.

In §4.1.2 the important ingredient for this section (i.e lemma 3) is proved. This lemma gives (under suitable and natural assumptions) the analytic continuation of integrals over a chains in \mathbb{C}^r , identifies for each of them a possible first pole σ_0 and gives a *very precise* bound for its order ρ_0 . This precise bound is *crucial* in the proof of the main result of this paper that is the second part of theorem 3. To obtain these informations, we must be more precise than classical proofs. For this reason our proof is quite technical and uses in particular a double induction arguments.

In §4.1.3 we will prove our second important ingredient (i.e lemma 4). This lemma gives for a class of integrals over chains in \mathbb{C}^r , the top order term in the principal part at the first pole.

In §4.1.1 we will prove two elementary but useful lemmas (lemmas 1 and 2) which will help us justify the convergence of integrals over chains of \mathbb{C}^r appearing in our proofs.

4.1 Four lemmas and their proofs

4.1.1 Two elementary lemmas (i.e lemmas 1 and 2)

Lemma 1. Let $\boldsymbol{\mu}^1, \dots, \boldsymbol{\mu}^k$ be k vectors of \mathbb{R}^r and let $l \in \mathbb{R}$. Set for all $\tau \in \mathbb{R}$:

$$F_r(\tau) := \int_{\mathbb{R}^r} \prod_{i=1}^r (1 + |y_i|)^l e^{\frac{\pi}{2} (|\tau| - \sum_{i=1}^r |y_i| - \sum_{i=1}^k |\langle \boldsymbol{\mu}^i, \mathbf{y} \rangle| - |\tau - \sum_{i=1}^r y_i - \sum_{i=1}^k \langle \boldsymbol{\mu}^i, \mathbf{y} \rangle|)} dy_1 \dots dy_r.$$

Then F_r has moderate growth in τ . More precisely, there exist $A = A(r), B = B(r)$ and $C = C(l, r) > 0$ such that: $\forall \tau \in \mathbb{R} \quad F_r(\tau) \leq C (1 + |\tau|)^{A|l|+B}$.

Remark: For $r = 1$ a more precise version of lemma 1 can be found in ([18], lemma 2).

Proof of Lemma 1:

Set $\psi_r(l; \tau) := \int_{\mathbb{R}^r} \prod_{i=1}^r (1 + |y_i|)^l e^{\frac{\pi}{2} (|\tau| - \sum_{i=1}^r |y_i| - |\tau - \sum_{i=1}^r y_i|)} dy_1 \dots dy_r$. Since

$$\sum_{i=1}^k |\langle \boldsymbol{\mu}^i, \mathbf{y} \rangle| + \left| \tau - \sum_{i=1}^r y_i - \sum_{i=1}^k \langle \boldsymbol{\mu}^i, \mathbf{y} \rangle \right| \geq \left| \sum_{i=1}^k \langle \boldsymbol{\mu}^i, \mathbf{y} \rangle + \tau - \sum_{i=1}^r y_i - \sum_{i=1}^k \langle \boldsymbol{\mu}^i, \mathbf{y} \rangle \right| \geq \left| \tau - \sum_{i=1}^r y_i \right|.$$

It follows that $F_r(\tau) \leq \psi_r(l; \tau)$. So to prove the lemma it suffices to prove the asserted bound for $\psi_r(l; \tau)$. We do this by induction on r .

• $r = 1$:

It suffices to prove the inequality for $\psi_1^+(l; \tau) = \int_0^{+\infty} (1 + y)^l e^{\frac{\pi}{2} (|\tau| - y - |\tau - y|)} dy$ and $\tau > 1$.

For $\tau > 1$ we have:

$$\begin{aligned} \psi_1^+(l; \tau) &= \int_0^{+\infty} (1 + y)^l e^{\frac{\pi}{2} (\tau - y - |\tau - y|)} dy \\ &= \int_0^\tau (1 + y)^l e^{\frac{\pi}{2} (\tau - y - |\tau - y|)} dy + \int_\tau^{+\infty} (1 + y)^l e^{\frac{\pi}{2} (\tau - y - |\tau - y|)} dy \\ &\ll_l \int_0^\tau (1 + y)^l dy + e^{\pi\tau} \int_\tau^{+\infty} (1 + y)^l e^{-\pi y} dy \ll_l (1 + \tau)^{l+1} + \int_1^{+\infty} (t + \tau)^l e^{-\pi t} dt \\ &\ll_l (1 + \tau)^{l+1} + (\tau + 1)^l \int_1^{+\infty} (1 + t)^l e^{-\pi t} dt \ll_l (1 + \tau)^{l+1}. \end{aligned}$$

• $r \geq 2$:

Assume that the lemma is true for $r - 1$. Thus there exist $A = A(r - 1)$ and $B = B(r - 1) > 0$ such that $\psi_{r-1}(l; \tau) \ll_l (1 + |\tau|)^{A|l|+B}$ ($\tau \in \mathbb{R}$). It follows that we have uniformly in $\tau \in \mathbb{R}$:

$$\begin{aligned} \psi_r(l; \tau) &:= \int_{\mathbb{R}^r} \prod_{i=1}^r (1 + |y_i|)^l e^{\frac{\pi}{2} (|\tau| - \sum_{i=1}^r |y_i| - |\tau - \sum_{i=1}^r y_i|)} dy_1 \dots dy_r \\ &= \int_{\mathbb{R}} \psi_{r-1}(l; \tau - y_r) (1 + |y_r|)^l e^{\frac{\pi}{2} (|\tau| - |\tau - y_r| - |y_r|)} dy_r \\ &\ll_{l,r} \int_{\mathbb{R}} (1 + |\tau - y_r|)^{A|l|+B} (1 + |y_r|)^l e^{\frac{\pi}{2} (|\tau| - |\tau - y_r| - |y_r|)} dy_r \quad (\text{by the induction hypothesis}) \\ &\ll_{l,r} (1 + |\tau|)^{A|l|+B} \int_{\mathbb{R}} (1 + |y_r|)^{(A+1)|l|+B} e^{\frac{\pi}{2} (|\tau| - |y_r| - |\tau - y_r|)} dy_r \\ &\ll_{l,r} (1 + |\tau|)^{A|l|+B} \psi_1((A+1)|l| + B; \tau). \end{aligned}$$

We complete the proof by using the preceding estimate when $r = 1$. \diamond

Lemma 2. Let $p \in \mathbb{R}$, $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{R}_+^{*n}$ and $\varepsilon > 0$ verifying $0 < \varepsilon < \inf_{i=1, \dots, r} a_i$. Set $W(s; \mathbf{z}) = W(s; z_1, \dots, z_r) := \Gamma(s - p - z_1 - \dots - z_r) \Gamma(s)^{-1} \left(\prod_{i=1}^r \Gamma(a_i + z_i) \right)$. Then $(s; \mathbf{z}) \mapsto W(s; \mathbf{z})$ is holomorphic in the set

$$\{(s, \mathbf{z}) = (\sigma + i\tau, \mathbf{x} + i\mathbf{y}) \in \mathbb{C} \times \mathbb{C}^r \mid \sigma > p - r\varepsilon \text{ and } |\Re(z_i)| < \varepsilon \forall i = 1, \dots, r\}$$

in which it satisfies the estimate:

$$\begin{aligned} W(s; \mathbf{z}) = W(\sigma + i\tau; \mathbf{x} + i\mathbf{y}) &\ll_{\sigma, p, \mathbf{a}, \varepsilon} (1 + |\tau|)^{2|\sigma| + |p| + r\varepsilon + 1} \prod_{i=1}^r (1 + |y_i|)^{|\sigma| + |p| + a_i + (r+1)\varepsilon + 1} \\ &\times e^{\frac{\pi}{2}(|\tau| - \sum_{i=1}^r |y_i| - |\tau - \sum_{i=1}^r y_i|)}. \end{aligned}$$

Proof of lemma 2:

It is well known that the Euler function $z \mapsto \Gamma(z)$ is holomorphic and has no zeros in the half-plane $\{z \in \mathbb{C} \mid \Re(z) > 0\}$. Moreover for any x_1, x_2 verifying $x_2 > x_1 > 0$ we have uniformly in $x \in [x_1, x_2]$ and $y \in \mathbb{R}$:

$$|\Gamma(x + iy)| = \sqrt{2\pi}(1 + |y|)^{x-1/2} e^{-\pi|y|/2} (1 + O_{x_1, x_2}(|y|^{-1})) \text{ as } |y| \rightarrow \infty. \quad (10)$$

We deduce that $(s, \mathbf{z}) \mapsto W(s, \mathbf{z})$ is holomorphic in

$$\{(s, \mathbf{z}) = (\sigma + i\tau, \mathbf{x} + i\mathbf{y}) \in \mathbb{C} \times \mathbb{C}^r \mid \sigma > p - r\varepsilon \text{ and } |\Re(z_i)| < \varepsilon \forall i = 1, \dots, r\}$$

in which it satisfies the estimate:

$$\begin{aligned} W(s; \mathbf{z}) &\ll_{\sigma, p, \mathbf{a}, \varepsilon} (1 + |\tau - y_1 - \dots - y_r|)^{\sigma - p - x_1 - \dots - x_r - \frac{1}{2}} (1 + |\tau|)^{-\sigma + \frac{1}{2}} \prod_{i=1}^r (1 + |y_i|)^{a_i + x_i - \frac{1}{2}} \\ &\times e^{\frac{\pi}{2}(|\tau| - \sum_{i=1}^r |y_i| - |\tau - \sum_{i=1}^r y_i|)} \\ &\ll_{\sigma, p, \mathbf{a}, \varepsilon} (1 + |\tau|)^{2|\sigma| + |p| + r\varepsilon + 1} \prod_{i=1}^r (1 + |y_i|)^{|\sigma| + |p| + a_i + (r+1)\varepsilon + 1} \times e^{\frac{\pi}{2}(|\tau| - \sum_{i=1}^r |y_i| - |\tau - \sum_{i=1}^r y_i|)}. \end{aligned}$$

This end the proof of lemma 2. \diamond

4.1.2 First crucial Lemma: Lemma 3

Before stating the lemma, we first introduce some needed notations:

Let $\sigma_1 \in \mathbb{R}_+^*$, $q \in \mathbb{N}$, $r \in \mathbb{N}$ and $\boldsymbol{\phi} \in \mathbb{R}^r$.

Let I be a finite subset of $\mathbb{R}^r \setminus \{\mathbf{0}\}$ and $\mathbf{u} = (u(\boldsymbol{\alpha}))_{\boldsymbol{\alpha} \in I}$ be a vector of positive integers.

Set for all $\delta', \varepsilon' > 0$,

$$\mathcal{D}_r(\delta'; \varepsilon') := \{(s, \mathbf{z}) \in \mathbb{C} \times \mathbb{C}^r \mid \sigma > \sigma_1 - \delta' \text{ and } |\Re(z_i)| < \varepsilon' \forall i = 1, \dots, r\}.$$

Let $\varepsilon, \delta > 0$. Let $L(s; \mathbf{z})$ be a holomorphic function on $\mathcal{D}_r(2\delta; 2\varepsilon)$.

Assume that there exist $A, B, w > 0$ and $\boldsymbol{\mu}^1, \dots, \boldsymbol{\mu}^p$ vectors of \mathbb{R}^r such that we have uniformly in $(s; \mathbf{z}) = (\sigma + i\tau, \mathbf{x} + i\mathbf{y}) \in \mathcal{D}_r(2\delta; 2\varepsilon)$:

$$\begin{aligned} L(s; \mathbf{z}) &\ll_{\sigma, \mathbf{x}} \prod_{i=1}^r (1 + |y_i|)^{A|\sigma| + B} (1 + |\tau|)^{A|\sigma| + B} \\ &\times e^{\frac{\pi}{2}(|w\tau| - \sum_{i=1}^r |y_i| - \sum_{i=1}^p |\langle \boldsymbol{\mu}^i, \mathbf{y} \rangle| - |w\tau - \sum_{i=1}^r y_i - \sum_{i=1}^p \langle \boldsymbol{\mu}^i, \mathbf{y} \rangle|)}. \end{aligned} \quad (11)$$

We denote by I_0 the set of real numbers which are the coordinates of at least one element of I , and by $\mathbb{Q}(I_0)$ the field generated by I_0 over \mathbb{Q} .

Set finally for all $\boldsymbol{\rho} = (\rho_1, \dots, \rho_r) \in]-\varepsilon, +\varepsilon[^n$ such that ρ_1, \dots, ρ_r are $\mathbb{Q}(I_0)$ -linearly independent,

$$T_r(s) := \frac{1}{(2\pi i)^r} \int_{\rho_1 - i\infty}^{\rho_1 + i\infty} \cdots \int_{\rho_r - i\infty}^{\rho_r + i\infty} \frac{L(s; \mathbf{z}) dz_1 \dots dz_r}{(s - \sigma_1 - \langle \boldsymbol{\phi}, \mathbf{z} \rangle)^q \prod_{\boldsymbol{\alpha} \in I} \langle \boldsymbol{\alpha}, \mathbf{z} \rangle^{u(\boldsymbol{\alpha})}}. \quad (12)$$

Lemma 1 imply that $s \mapsto T_r(s)$ converges absolutely in $\{\sigma > \sigma_1 + \sum_{i=1}^n |\phi_i \rho_i|\}$.

The key lemma of this paper is the following:

Lemma 3. *There exists $\eta > 0$ such that $s \mapsto T_r(s)$ has a meromorphic continuation with moderate growth to the half-plane $\{\Re(s) > \sigma_1 - \eta\}$ with at most a single pole at $s = \sigma_1$.*

If $s = \sigma_1$ is a pole of $T_r(s)$ then its order is at most

$$d_r := \sum_{\boldsymbol{\alpha} \in I} u(\boldsymbol{\alpha}) - \text{rank}(I) + q - \varepsilon_0(\boldsymbol{\phi}; L),$$

Where the index $\varepsilon_0(\boldsymbol{\phi}; L)$ is defined by:

1. $\varepsilon_0(\boldsymbol{\phi}; L) = 1$ if $\boldsymbol{\phi} \in \text{con}^*(I) \setminus \{\mathbf{0}\}$ and if there exist two analytic functions K and W such that for all $(s, \mathbf{z}) \in \mathcal{D}(2\delta; 2\varepsilon)$ $L(s; \mathbf{z}) = K(s; \mathbf{z})W(s; \langle \boldsymbol{\alpha}, \mathbf{z} \rangle_{\boldsymbol{\alpha} \in I})$ and $W(s; \mathbf{0}) \equiv 0$;
2. $\varepsilon_0(\boldsymbol{\phi}; L) = 0$ otherwise.

Proof of lemma 3:

We proceed by induction on r .

Throughout the discussion, we use the following notations. Given $\mathbf{z} = (z_1, \dots, z_r)$ we set: $\mathbf{z}' := (z_1, \dots, z_{r-1})$ and $l(\mathbf{z}) := \frac{1}{z_r} \mathbf{z}' = \left(\frac{z_1}{z_r}, \dots, \frac{z_{r-1}}{z_r} \right)$ if $z_r \neq 0$.

Step 1: Proof when $r = 1$:

Set $A = \prod_{\boldsymbol{\alpha} \in I} \alpha^{u(\boldsymbol{\alpha})}$ and $c = \sum_{\boldsymbol{\alpha} \in I} u(\boldsymbol{\alpha})$. We have:

$$T_1(s) = \frac{1}{(2\pi i)} \int_{\rho_1 - i\infty}^{\rho_1 + i\infty} \frac{L(s; z) dz}{(s - \sigma_1 - \phi z)^q \prod_{\boldsymbol{\alpha} \in I} (\alpha z)^{u(\boldsymbol{\alpha})}} = \frac{1}{(2\pi i)A} \int_{\rho_1 - i\infty}^{\rho_1 + i\infty} \frac{L(s; z) dz}{(s - \sigma_1 - \phi z)^q z^c}.$$

From our assumptions (see (11)) and lemma 1 it follows that $s \mapsto T_1(s)$ converges absolutely and defines a holomorphic function with moderate growth in $\{\sigma > \sigma_1 - \eta\}$ where $\eta = \inf(-\phi \rho_1, \delta)$.

- If $\phi \rho_1 < 0$ then $\eta > 0$. This proves the lemma in this case.
- if $\phi = 0$ then $T_1(s) = \frac{1}{(2\pi i)A(s - \sigma_1)^q} \int_{\rho_1 - i\infty}^{\rho_1 + i\infty} \frac{L(s; z) dz}{z^c}$. It follows also from Lemma 1 and (11) that $s \mapsto T_1(s)$ has a meromorphic continuation with moderate growth to the half-plane $\{\sigma > \sigma_1 - \delta\}$ with at most one pole at $s = \sigma_1$ of order at most $q \leq d_1$.
- We assume $\phi \rho_1 > 0$:

The residue theorem and lemma 1 imply that for $\sigma > \sigma_1 + \phi \rho_1$:

$$T_1(s) = T_1'(s) + T_1''(s) \text{ where } T_1'(s) = \frac{1}{(2\pi i)A} \int_{-\rho_1 - i\infty}^{-\rho_1 + i\infty} \frac{L(s; z) dz}{(s - \sigma_1 - \phi z)^q z^c} \text{ and } T_1''(s) = \frac{1}{A} \text{Res}_{z=0} \frac{L(s; z)}{(s - \sigma_1 - \phi z)^q z^c}.$$

Lemma 1 and (11) imply that $s \mapsto T_1'(s)$ converges absolutely and defines a holomorphic function with moderate growth in the half-plane $\{\sigma > \sigma_1 - \eta\}$ where $\eta = \inf(\delta, \phi \rho_1)$.

If $c = 0$ then $T_1''(s) \equiv 0$. Thus $T_1(s) = T_1'(s)$ satisfies the conclusions of lemma 3.

We assume now that $c \geq 1$. An easy computation shows that

$$T_1''(s) = \frac{1}{A} \sum_{k=0}^{c-1} \frac{\binom{-q}{k} (-\phi)^k}{(c-1-k)!} \frac{\partial_z^{(c-1-k)} L(s; 0)}{(s - \sigma_1)^{q+k}}.$$

We deduce that $T_1(s)$ has a meromorphic continuation with moderate growth to $\{\sigma > \sigma_1 - \eta\}$ with at most one pole at $s = \sigma_1$ of order at most:

1. $\text{ord}_{s=\sigma_1} T_1(s) \leq q + c - 1 = q + \sum_{\alpha \in I} u(\alpha) - \text{rank}(I) = d_1$ if $L(s, 0) \neq 0$;
2. $\text{ord}_{s=\sigma_1} T_1(s) \leq q + c - 2 = q + \sum_{\alpha \in I} u(\alpha) - \text{rank}(I) - 1 \leq d_1$ if $L(s, 0) = 0$;

This proves Lemma 3 if $r = 1$.

Step 2: Let $r \geq 2$. We assume that lemma 3 is true for any $r' \leq r - 1$. We will show that it also remains true for r :

We justify this assertion by induction on the integer

$$h = h(\phi, \rho) := \#\{i \in \{1, \dots, r\} \mid \phi_i \rho_i \geq 0\} \in \{0, \dots, r\}.$$

• **Proof of lemma 3 for $h = 0$:**

Since $h = 0$ then for each $i = 1, \dots, r$, $\phi_i \rho_i < 0$. It follows from lemma 1 and (11) that $s \mapsto T_r(s)$ converges absolutely and defines a holomorphic function with moderate growth in the half-plane $\{\sigma > a - \eta\}$ where $\eta = \inf(-\langle \phi, \rho \rangle, \delta) > 0$. Thus lemma 3 is also true in this case.

• Let $h \in \{1, \dots, r\}$. We assume that lemma 3 is true for $h(\phi, \rho) \leq h - 1$. We will prove that it remains true for $h(\phi, \rho) = h$:

If $\phi = \mathbf{0}$ then $T_r(s) = \frac{1}{(2\pi i)^r (s - \sigma_1)^q} \int_{\rho_1 - i\infty}^{\rho_1 + i\infty} \dots \int_{\rho_r - i\infty}^{\rho_r + i\infty} \frac{L(s; \mathbf{z}) dz_1 \dots dz_r}{\prod_{\alpha \in I} \langle \alpha, \mathbf{z} \rangle^{u(\alpha)}}$. Since the ρ_i are linearly independent over $\mathbb{Q}(I_0)$, lemma 1 and (11) imply that lemma 3 is true in this case, in the sense that there is at most one pole at $s = \sigma_1$ of order at most $q \leq d_r$.

If $\phi \neq \mathbf{0}$ and $\phi_i \rho_i \leq 0$ for all $i = 1, \dots, r$, then there exists i_0 such that $\phi_{i_0} \rho_{i_0} < 0$. In this case, it is also easy to see that $s \mapsto T_r(s)$ is *holomorphic* with moderate growth in $\{\sigma > \sigma_1 - \eta\}$ where $\eta = \inf(\delta, -\phi_{i_0} \rho_{i_0}) > 0$. It follows that lemma 3 is also true in this case.

So to finish the proof of lemma 3 it suffices to consider the case where there exists $i \in \{1, \dots, r\}$ such that $\phi_i \rho_i > 0$.

Without loss of generality we can assume that $\phi_r \rho_r > 0$.

$$\text{Set } J := \left\{ \alpha \in I \mid \alpha_r \neq 0 \text{ and } \left| \frac{\alpha_1 \rho_1 + \dots + \alpha_{r-1} \rho_{r-1}}{\alpha_r} \right| < |\rho_r| \right\}.$$

Consider the equivalence relation \mathcal{R} defined on J by: $\alpha \mathcal{R} \gamma$ iff $\frac{1}{\alpha_r} \alpha' = \frac{1}{\gamma_r} \gamma'$ iff $l(\alpha) = l(\gamma)$.

It's clear that $\alpha \mathcal{R} \gamma$ iff $\alpha_r \gamma = \gamma_r \alpha$. Denote by J_1, \dots, J_t the equivalence classes of \mathcal{R} (they form a partition of J).

Choose for each $k = 1, \dots, t$, an element $\alpha^k \in J_k$ and set $c_k := \sum_{\alpha \in J_k} u(\alpha)$.

Fix now $s = \sigma + i\tau \in \mathbb{C}$ and $\mathbf{z}' = (z_1, \dots, z_{r-1}) \in \mathbb{C}^{r-1}$ such that $\sigma > \sigma_1 + \sum_{i=1}^r |\phi_i \rho_i|$ and $\forall i = 1, \dots, r-1$, $\Re(z_i) = \rho_i$.

The function

$$\begin{aligned} z_r \mapsto I(z_r) &:= \frac{L(s; \mathbf{z})}{(s - \sigma_1 - \langle \phi, \mathbf{z} \rangle)^q \prod_{\alpha \in I} \langle \alpha, \mathbf{z} \rangle^{u(\alpha)}} \\ &= \frac{L(s; \mathbf{z}', z_r)}{(s - \sigma_1 - \langle \phi', \mathbf{z}' \rangle - \phi_r z_r)^q \prod_{\alpha \in I} (\langle \alpha', \mathbf{z}' \rangle + \alpha_r z_r)^{u(\alpha)}} \end{aligned}$$

is meromorphic in $\{z_r \in \mathbb{C} \mid |\Re(z_r)| < \rho_r\}$ with at most poles at the points $z_r = -\frac{1}{\alpha_r} \langle \alpha', \mathbf{z}' \rangle = -\langle l(\alpha), \mathbf{z}' \rangle$ ($\alpha \in J$). Moreover since ρ_1, \dots, ρ_r are $\mathbb{Q}(I_0)$ -linearly independent, it follows that $\langle l(\alpha), \mathbf{z}' \rangle = \langle l(\gamma), \mathbf{z}' \rangle$ iff $l(\alpha) = l(\gamma)$ iff $\alpha \mathcal{R} \gamma$.

Therefore the poles of $z_r \mapsto I(z_r)$ in $\{z_r \in \mathbb{C} \mid |\Re(z_r)| < \rho_r\}$ are the points

$$z_r = -\langle l(\alpha^k), \mathbf{z}' \rangle \quad (k = 1, \dots, t)$$

and for any k , $ord_{z_r = -\langle l(\alpha^k), \mathbf{z}' \rangle} I(z_r) = \sum_{\alpha \in J_k} u(\alpha) = c_k$.

The residue theorem, (11) and lemma 1 imply then that there exist constants $A_1, \dots, A_t \in \mathbb{R}$ such that:

$$\forall \sigma > \sigma_1 + \sum_{i=1}^r |\phi_i \rho_i|, \quad T_r(s) = T_r^0(s) + \sum_{k=1}^t A_k T_{r-1}^k(s) \quad (13)$$

where

$$T_r^0(s) := \frac{1}{(2\pi i)^r} \int_{\rho_1 - i\infty}^{\rho_1 + i\infty} \cdots \int_{\rho_{r-1} - i\infty}^{\rho_{r-1} + i\infty} \int_{-\rho_r - i\infty}^{-\rho_r + i\infty} \frac{L(s; \mathbf{z}) dz_1 \dots dz_r}{(s - \sigma_1 - \langle \phi, \mathbf{z} \rangle)^q \prod_{\alpha \in I} \langle \alpha, \mathbf{z} \rangle^{u(\alpha)}}$$

and for each $k = 1, \dots, t$:

$$T_{r-1}^k(s) := \frac{1}{(2\pi i)^{r-1}} \int_{\rho_1 - i\infty}^{\rho_1 + i\infty} \cdots \int_{\rho_{r-1} - i\infty}^{\rho_{r-1} + i\infty} \left(\frac{\partial}{\partial z_r} \right)^{c_k - 1} N(s; \mathbf{z}', -\langle l(\alpha^k), \mathbf{z}' \rangle) dz_1 \dots dz_{r-1},$$

$$\text{where } N(s; \mathbf{z}', z_r) = N(s; (\mathbf{z}', z_r)) = N(s; \mathbf{z}) := \frac{L(s; \mathbf{z})}{(s - \sigma_1 - \langle \phi, \mathbf{z} \rangle)^q \prod_{\alpha \in I \setminus J_k} \langle \alpha, \mathbf{z} \rangle^{u(\alpha)}}.$$

Since $h(\phi, (\rho_1, \dots, \rho_{r-1}, -\rho_r)) = h(\phi, \rho) - 1 = h - 1$, the induction hypothesis for $h - 1$ implies that $s \mapsto T_r^0(s)$ satisfies the conclusions of lemma 3. So to conclude, it is enough to prove lemma 3 for each $s \mapsto T_{r-1}^k(s)$.

We then choose and fix any $k \in \{1, \dots, t\}$ for the rest of the discussion.

An easy computation shows that:

$$T_{r-1}^k(s) = \sum_{u+v+\sum_{\alpha \in I \setminus J_k} k_\alpha = c_k - 1} w(u, v, (k_\alpha)) R_k(u, v, (k_\alpha); s) \quad (14)$$

where u, v and the k_α are in \mathbb{N}_0 , each $w(u, v, (k_\alpha)) \in \mathbb{R}$ and

$$R_k(u, v, (k_\alpha); s) := \frac{1}{(2\pi i)^{r-1}} \int_{\rho_1 - i\infty}^{\rho_1 + i\infty} \cdots \int_{\rho_{r-1} - i\infty}^{\rho_{r-1} + i\infty} \frac{\frac{\partial^u L}{\partial z_r^u}(s; \mathbf{z}', -\langle l(\alpha^k), \mathbf{z}' \rangle) dz_1 \dots dz_{r-1}}{(s - \sigma_1 - \langle \phi' - \phi_r l(\alpha^k), \mathbf{z}' \rangle)^{q+v} \prod_{\alpha \in I \setminus J_k} \langle \alpha' - \alpha_r l(\alpha^k), \mathbf{z}' \rangle^{u(\alpha) + k_\alpha}}. \quad (15)$$

So to conclude it suffices to prove lemma 3 for each $R_k(u, v, (k_\alpha); s)$.

We fix now $u, v \in \mathbb{N}_0$ and $(k_\alpha)_{\alpha \in I \setminus J_k}$ such that

$$u + v + \sum_{\alpha \in I \setminus J_k} k_\alpha = c_k - 1. \quad (16)$$

It is clear that there exist $\delta, \varepsilon > 0$ such that the function $(s; \mathbf{z}') \mapsto \frac{\partial^u L}{\partial z_r^u}(s; \mathbf{z}', -\langle l(\alpha^k), \mathbf{z}' \rangle)$ is holomorphic in $\mathcal{D}_{r-1}(2\delta, 2\varepsilon)$ and satisfies an estimate similar to (11) (the last assertion is justified by using Cauchy's integral formula).

The induction hypothesis on r implies that there exists $\eta > 0$ such that $s \mapsto R_k(u, v, (k_\alpha); s)$ has meromorphic continuation with moderate growth to the half-plane $\{\sigma > \sigma_1 - \eta\}$ with at most one pole at $s = \sigma_1$ of order at most

$$ord_{s=\sigma_1} R_k(u, v, (k_\alpha); s) \leq \left(\sum_{\alpha \in I \setminus J_k} u(\alpha) + k_\alpha \right) - \text{rank}(V) + (q+v) - \varepsilon_0 \left(\phi' - \phi_r l(\alpha^k); \tilde{L}_u \right), \quad (17)$$

where $V := \{\alpha' - \alpha_r l(\alpha^k) \mid \alpha \in I \setminus J_k\}$ and $\tilde{L}_u(s; \mathbf{z}') := \frac{\partial^u L}{\partial z_r^u}(s; \mathbf{z}', -\langle l(\alpha^k), \mathbf{z}' \rangle)$.

Set $\tilde{V} := \left\{ \alpha - \frac{\alpha_r}{\alpha_r^k} \alpha^k \mid \alpha \in I \setminus J_k \right\} = \{(\alpha' - \alpha_r l(\alpha^k), 0) \mid \alpha \in I \setminus J_k\} = V \times \{\mathbf{0}\}$.

It is clear that $\text{rank}(\tilde{V}) = \text{rank}(V)$. Moreover it follows from the definition of α^k that $\alpha_r^k \neq 0$, and therefore $\alpha^k \notin \text{Vect}_{\mathbb{R}}(\tilde{V})$. We deduce that:

$$\text{rank}(I) = \text{rank}(\tilde{V} \cup \{\alpha^k\}) = \text{rank}(\tilde{V}) + 1 = \text{rank}(V) + 1. \quad (18)$$

So it follows from (17), (16) and (18) that:

$$\begin{aligned} \text{ord}_{s=\sigma_1} R_k(u, v, (k_{\alpha}); s) &\leq \left(\sum_{\alpha \in I \setminus J_k} u(\alpha) + k_{\alpha} \right) - \text{rank}(V) + q + v - \varepsilon_0 \left(\phi' - \phi_r l(\alpha^k); \tilde{L}_u \right) \\ &= c_k - 1 + \sum_{\alpha \in I \setminus J_k} u(\alpha) - \text{rank}(V) + q - u - \varepsilon_0 \left(\phi' - \phi_r l(\alpha^k); \tilde{L}_u \right) \\ &\leq (c_k + \sum_{\alpha \in I \setminus J_k} u(\alpha)) - (\text{rank}(V) + 1) + q - \varepsilon_0 \left(\phi' - \phi_r l(\alpha^k); \tilde{L}_u \right) - u \\ &\leq \sum_{\alpha \in I} u(\alpha) - \text{rank}(I) + q - \varepsilon_0 \left(\phi' - \phi_r l(\alpha^k); \tilde{L}_u \right) - u. \end{aligned}$$

Thus from the definition of d_r given above we see that:

$$\text{ord}_{s=\sigma_1} R_k(u, v, (k_{\alpha}); s) \leq d_r + \varepsilon_0(\phi; L) - \varepsilon_0 \left(\phi' - \phi_r l(\alpha^k); \tilde{L}_u \right) - u. \quad (19)$$

If $\varepsilon_0(\phi; L) - \varepsilon_0 \left(\phi' - \phi_r l(\alpha^k); \tilde{L}_u \right) \leq 0$ or $u \neq 0$, then $\text{ord}_{s=\sigma_1} R_k(u, v, (k_{\alpha}); s) \leq d_r$. This finishes the proof of the lemma in all cases except the following possibility:

$u = 0$, $\varepsilon_0(\phi; L) = 1$ and $\varepsilon_0 \left(\phi' - \phi_r l(\alpha^k); \tilde{L}_0 \right) = 0$.

Our hypothesis about L evidently implies that \tilde{L}_0 has the same form:

$$\tilde{L}_0(s, \mathbf{z}') = \tilde{K}(s; \mathbf{z}') \tilde{W}(s; \langle \boldsymbol{\mu}, \mathbf{z}' \rangle_{\boldsymbol{\mu} \in V}) \text{ with } \tilde{W}(s; \mathbf{0}) \equiv 0. \quad (20)$$

This and the fact that $\varepsilon_0 \left(\phi' - \phi_r l(\alpha^k); \tilde{L}_0 \right) = 0$ now imply that we must have:

$$\phi' - \phi_r l(\alpha^k) = \phi' - \frac{\phi_r}{\alpha_r^k} (\alpha^k)' = (\phi_1, \dots, \phi_{r-1}) - \frac{\phi_r}{\alpha_r^k} (\alpha_1^k, \dots, \alpha_{r-1}^k) \notin \text{con}^*(V) \setminus \{\mathbf{0}\}. \quad (21)$$

However, it follows from the equality $\varepsilon_0(\phi; L) = 1$ that $\phi \in \text{con}^*(I) \setminus \{\mathbf{0}\}$. Thus, there exists $\{\lambda_{\alpha}\}_{\alpha \in I} \subset \mathbb{R}_+^*$ such that $\phi = \sum_{\alpha \in I} \lambda_{\alpha} \alpha$. This implies $\phi' = \sum_{\alpha \in I} \lambda_{\alpha} \alpha'$ and $\phi_r = \sum_{\alpha \in I} \lambda_{\alpha} \alpha_r$. We deduce that:

$$\begin{aligned} \phi' - \phi_r l(\alpha^k) &= \sum_{\alpha \in I} \lambda_{\alpha} \alpha' - \sum_{\alpha \in I} \lambda_{\alpha} \alpha_r l(\alpha^k) = \sum_{\alpha \in I} \lambda_{\alpha} (\alpha' - \alpha_r l(\alpha^k)) \\ &= \sum_{\alpha \in I \setminus J_k} \lambda_{\alpha} (\alpha' - \alpha_r l(\alpha^k)) \text{ (because } \alpha' - \alpha_r l(\alpha^k) = 0 \text{ if } \alpha \in J_k). \end{aligned}$$

Thus, $\phi' - \phi_r l(\alpha^k) \in \text{con}^*(V)$, which, thanks to (21), now gives $\phi' - \phi_r l(\alpha^k) = \mathbf{0}$.

From lemma 1 and the expressions (15) and (16) it follows then that:

$$\text{ord}_{s=\sigma_1} R_k(0, v, (k_{\alpha}); s) \leq q + v = q + c_k - 1 - \sum_{\alpha \in I \setminus J_k} k_{\alpha} \leq q + \sum_{\alpha \in J_k} u(\alpha) - 1. \quad (22)$$

But $\phi' - \phi_r l(\alpha^k) = \mathbf{0}$ implies also that $\phi - \frac{\phi_r}{\alpha_r^k} \alpha^k = (\phi' - \phi_r l(\alpha^k), 0) = \mathbf{0}$. Therefore we get $\phi = \frac{\phi_r}{\alpha_r^k} \alpha^k$. Since $\phi \neq \mathbf{0}$, we conclude that $\phi_r \neq 0$. Thus $\mathbb{R}\alpha^k = \mathbb{R}\phi = \text{Vect}_{\mathbb{R}}(J_k)$. This and the fact that $\phi \in \text{con}^*(I)$ imply then that $\text{rank}(I) = \text{rank}(I \setminus J_k)$, unless $I = J_k$.

Assume that $I \neq J_k$:

In this event, combining (22) with the fact that $\text{rank}(I \setminus J_k) = \text{rank}(I)$, we conclude:

$$\begin{aligned} \text{ord}_{s=\sigma_1} R_k(0, v, (k_\alpha); s) &\leq q + \sum_{\alpha \in I} u(\alpha) - \text{rank}(I) - \sum_{\alpha \in I \setminus J_k} u(\alpha) + \text{rank}(I) - 1 \\ &\leq q + \sum_{\alpha \in I} u(\alpha) - \text{rank}(I) - 1 - \left(\sum_{\alpha \in I \setminus J_k} u(\alpha) - \text{rank}(I \setminus J_k) \right) \\ &\leq q + \sum_{\alpha \in I} u(\alpha) - \text{rank}(I) - 1 - (\#(I \setminus J_k) - \text{rank}(I \setminus J_k)) \\ &\leq q + \sum_{\alpha \in I} u(\alpha) - \text{rank}(I) - 1 \leq d_r. \end{aligned}$$

Assume that $I = J_k$:

It is then clear that $V = \emptyset$ (see (17)). In this case, (20) then implies that $\tilde{L}_0(s; \mathbf{z}') \equiv 0$, and (15) implies that $R_k(0, v, (k_\alpha); s) \equiv 0$. So, it is obvious that $\text{ord}_{s=\sigma_1} R_k(0, v, (k_\alpha); s) \leq d_r$.

We conclude that for any $u, v, (k_\alpha)$, $\text{ord}_{s=\sigma_1} R_k(u, v, (k_\alpha); s) \leq d_r$. This finishes the induction argument on h , therefore, also on r , and completes the proof of lemma 3. \diamond

4.1.3 Second crucial lemma: Lemma 4

Lemma 4. *Let $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{R}_+^{*r}$ and $a = |\mathbf{a}| = a_1 + \dots + a_r$. Let I be a finite nonempty subset of $\mathbb{R}_+^r \setminus \{\mathbf{0}\}$, $\mathbf{u} = (u(\beta))_{\beta \in I}$ a vector of positive integers, and $\mathbf{h} = (h_1, \dots, h_r) \in \mathbb{R}_+^{*r}$. Assume that: $\mathbf{1} \in \text{con}(I)$ and that $\langle \beta, \mathbf{a} \rangle = 1$ for all $\beta \in I$.*

*Let $\rho \in \mathbb{R}_+^{*r}$. For $\sigma = \Re(s) > a + |\rho|$ set:*

$$\mathcal{R}(s) := \frac{1}{(2\pi i)^r} \int_{\rho_1 - i\infty}^{\rho_1 + i\infty} \dots \int_{\rho_r - i\infty}^{\rho_r + i\infty} \frac{\Gamma(s - a - z_1 - \dots - z_r) \prod_{i=1}^r \Gamma(a_i + z_i) dz}{\Gamma(s) \prod_{k=1}^r h_k^{a_k + z_k} \prod_{\beta \in I} \langle \beta, \mathbf{z} \rangle^{u(\beta)}}. \quad (23)$$

Then there exists $\eta > 0$ such that $s \mapsto \mathcal{R}(s)$ has a meromorphic continuation to the half-plane $\{\sigma > a - \eta\}$ with exactly one pole at $s = a$ of order $\rho_0 := \sum_{\beta \in I} u(\beta) - \text{rank}(I) + 1$. Moreover we

have $\mathcal{R}(s) \sim_{s \rightarrow a} \frac{A_0(I; \mathbf{u}; \mathbf{h})}{(s - a)^{\rho_0}}$, where $A_0(I; \mathbf{u}; \mathbf{h})$ is the volume constant (see §2.3.2) associated to I , \mathbf{u} and \mathbf{h} .

Proof of Lemma 4:

We fix $\rho \in \mathbb{R}_+^{*r}$. From lemma 2 and lemma 1 it follows easily that the integral $\mathcal{R}(s)$ converges for any sufficiently large σ .

Set $q := \sum_{\beta \in I} u(\beta)$ and define the vectors $\alpha^1, \dots, \alpha^q \in \mathbb{R}_+^r$ by:

$\{\alpha^i \mid i = 1, \dots, q\} = I$ and $\forall \beta \in I \# \{i \in \{1, \dots, q\} \mid \alpha^i = \beta\} = u(\beta)$ (i.e. the family of vectors $(\alpha^i)_{i=1, \dots, q}$ is obtained by repeating each vector $\beta \in I$ $u(\beta)$ times).

We then define $\mu^1, \dots, \mu^r \in \mathbb{R}_+^q$ by setting: $\mu_i^k = \alpha_k^i \forall i = 1, \dots, q$ and $\forall k = 1, \dots, r$.

Define the generalized polynomial with positive coefficients:

$$G(\mathbf{X}) = 1 + P_{(I; \mathbf{u}; \mathbf{h})}(\mathbf{X}) := 1 + \sum_{k=1}^r h_k \mathbf{X}^{\mu^k}.$$

We note that G depends on all the variables X_1, \dots, X_q since for any $i = 1, \dots, q$ there exists $k \in \{1, \dots, r\}$ such that $\mu_i^k = \alpha_k^i \neq 0$. We will first prove:

Claim: For $\sigma \gg 1$,

$$\mathcal{R}(s) = \int_{[1, +\infty[^q} G^{-s}(\mathbf{x}) \, d\mathbf{x}. \quad (24)$$

Proof of Claim: For $\sigma = \Re(s) \gg 1$ we have the following identities:

$$\begin{aligned} \mathcal{R}(s) &:= \frac{1}{(2\pi i)^r} \int_{\rho_1 - i\infty}^{\rho_1 + i\infty} \cdots \int_{\rho_r - i\infty}^{\rho_r + i\infty} \frac{\Gamma(s - a - z_1 - \cdots - z_r) \prod_{i=1}^r \Gamma(a_i + z_i) \, d\mathbf{z}}{\Gamma(s) \prod_{i=1}^r h_i^{a_i + z_i} \prod_{k=1}^q \langle \alpha^k, \mathbf{z} \rangle} \\ &= \frac{1}{(2\pi i)^r} \int_{a_1 + \rho_1 - i\infty}^{a_1 + \rho_1 + i\infty} \cdots \int_{a_r + \rho_r - i\infty}^{a_r + \rho_r + i\infty} \frac{\Gamma(s - z_1 - \cdots - z_r) \prod_{i=1}^r \Gamma(z_i) \, d\mathbf{z}}{\Gamma(s) \prod_{i=1}^r h_i^{z_i} \prod_{k=1}^q (\langle \alpha^k, \mathbf{z} \rangle - 1)} \\ &= \frac{1}{(2\pi i)^r} \int_{a_1 + \rho_1 - i\infty}^{a_1 + \rho_1 + i\infty} \cdots \int_{a_r + \rho_r - i\infty}^{a_r + \rho_r + i\infty} \Gamma(s - z_1 - \cdots - z_r) \prod_{i=1}^r \Gamma(z_i) \Gamma(s)^{-1} \prod_{i=1}^r h_i^{-z_i} \\ &\quad \times \left(\int_{[1, +\infty[^q} \prod_{k=1}^q x_k^{-\langle \alpha^k, \mathbf{z} \rangle} \, dx_1 \dots dx_q \right) dz_1 \dots dz_r \\ &= \frac{1}{(2\pi i)^r} \int_{a_1 + \rho_1 - i\infty}^{a_1 + \rho_1 + i\infty} \cdots \int_{a_r + \rho_r - i\infty}^{a_r + \rho_r + i\infty} \Gamma(s - z_1 - \cdots - z_r) \prod_{i=1}^r \Gamma(z_i) \Gamma(s)^{-1} \prod_{i=1}^r h_i^{-z_i} \\ &\quad \times \left(\int_{[1, +\infty[^q} \prod_{i=1}^r \mathbf{x}^{-z_i \boldsymbol{\mu}^i} \, dx_1 \dots dx_q \right) dz_1 \dots dz_r. \end{aligned} \quad (25)$$

In the other hand, we have uniformly in $\mathbf{z} \in \mathbb{C}^r$ such that $\Re(z_i) = a_i + \rho_i$:

$$\forall \mathbf{x} \in [1, +\infty[^q \quad \left| \prod_{i=1}^r \mathbf{x}^{-z_i \boldsymbol{\mu}^i} \right| = \left| \prod_{k=1}^q x_k^{-\langle \alpha^k, \mathbf{z} \rangle} \right| = \prod_{k=1}^q x_k^{-\langle \alpha^k, \mathbf{a} \rangle - \langle \alpha^k, \boldsymbol{\rho} \rangle} = \prod_{k=1}^q x_k^{-1 - \langle \alpha^k, \boldsymbol{\rho} \rangle}.$$

We deduce that the integral $\int_{[1, +\infty[^q} \prod_{i=1}^r \mathbf{x}^{-z_i \boldsymbol{\mu}^i} \, dx_1 \dots dx_q$ converges absolutely and uniformly in $\mathbf{z} \in \mathbb{C}^r$ such that $\Re(z_i) = a_i + \rho_i$. This, (25), lemma 2, and lemma 1 imply then that for $\sigma \gg 1$, we have

$$\begin{aligned} \mathcal{R}(s) &= \int_{[1, +\infty[^q} \left[\frac{1}{(2\pi i)^r} \int_{a_1 + \rho_1 - i\infty}^{a_1 + \rho_1 + i\infty} \cdots \int_{a_r + \rho_r - i\infty}^{a_r + \rho_r + i\infty} \Gamma(s - z_1 - \cdots - z_r) \prod_{i=1}^r \Gamma(z_i) \Gamma(s)^{-1} \right. \\ &\quad \left. \times \prod_{i=1}^r \left(h_i \mathbf{x}^{\boldsymbol{\mu}^i} \right)^{-z_i} dz_1 \dots dz_r \right] dx_1 \dots dx_q. \end{aligned}$$

Mellin's formula (9) implies then that for $\sigma \gg 1$, we have

$$\mathcal{R}(s) = \int_{[1, +\infty[^q} \left(1 + \sum_{i=1}^r h_i \mathbf{x}^{\boldsymbol{\mu}^i} \right)^{-s} dx_1 \dots dx_q = \int_{[1, +\infty[^n} G^{-s}(\mathbf{x}) \, d\mathbf{x}.$$

This end the proof of the claim.

So to conclude it suffices to check that $s \mapsto Y(G; s) := \int_{[1, +\infty[^q} G^{-s}(\mathbf{x}) \, d\mathbf{x}$ satisfies the assertions of Lemma 4.

Let $\mathcal{E}^\infty(G) = \text{conv}(\text{supp}(G) - \mathbb{R}_+^q)$ denote the Newton polyhedron at infinity of G . Denote by G_0 the smallest face that meets the diagonal. It follows from Sargos' result (see §2.3) that there exists $\eta > 0$ such that $Y(G; s)$ has a meromorphic continuation to the half-plane $\{\sigma > \sigma_0 - \eta\}$ (where

$\sigma_0 = \sigma_0(G)$ with moderate growth and exactly one pole at $s = \sigma_0$ of order $\rho_0 := \text{codim}G_0$. Moreover σ_0 is characterized geometrically by: $\sigma_0^{-1}\mathbf{1} = \Delta \cap G_0 = \Delta \cap \mathcal{E}^\infty(G)$ and $Y(G; s) \sim_{s \rightarrow \sigma_0} \frac{A_0(G)}{(s - \sigma_0)^{\rho_0}}$ where $A_0(G)$ is the volume constant associated to the polynomial G . It is easy to see that in our case $A_0(G)$ is equal to the volume constant (see §2.3.2) $A_0(I; \mathbf{u}; \mathbf{h})$ associated to I , \mathbf{u} and \mathbf{h} .

By our hypothesis, we have $\mathbf{1} \in \text{con}(I) = \text{con}(\{\boldsymbol{\alpha}^k \mid k = 1, \dots, q\})$. Thus there exists $\mathbf{v} = (v_1, \dots, v_q) \in \mathbb{R}_+^q \setminus \{\mathbf{0}\}$ such that $\mathbf{1} = \sum_{k=1}^q v_k \boldsymbol{\alpha}^k$. It follows that:

$$\forall i = 1, \dots, r \quad \langle \mathbf{v}, \boldsymbol{\mu}^i \rangle = \sum_{k=1}^q v_k \mu_k^i = \sum_{k=1}^q v_k \alpha_i^k = 1.$$

Since $\text{supp}(G) = \{\boldsymbol{\mu}^i \mid i = 1, \dots, r\} \cup \{\mathbf{0}\}$, we conclude that $L_{\mathbf{v}} := \{\mathbf{x} \in \mathbb{R}^q \mid \langle \mathbf{v}, \mathbf{x} \rangle = 1\}$ is a support plane of $\mathcal{E}^\infty(G)$. Thus:

$$F_{\mathbf{v}}^\infty := L_{\mathbf{v}} \cap \mathcal{E}^\infty(G) = \text{conv}(\{\boldsymbol{\mu}^i \mid i = 1, \dots, r\}) \text{ is a face of the polyhedron } \mathcal{E}^\infty(G). \quad (26)$$

By our hypothesis we know that $\sum_{k=1}^r a_k \mu_k^i = \langle \mathbf{a}, \boldsymbol{\alpha}^i \rangle = 1 \forall i = 1, \dots, q$, which implies $\frac{1}{a} \mathbf{1} = \sum_{k=1}^r \frac{a_k}{a} \boldsymbol{\mu}^k \in \text{conv}(\{\boldsymbol{\mu}^i \mid i = 1, \dots, r\})$. Thus, $\frac{1}{a} \mathbf{1} \in F_{\mathbf{v}}^\infty \cap \Delta$, that is, the face $F_{\mathbf{v}}^\infty$ must meet the diagonal at $\frac{1}{a} \mathbf{1}$. It follows that $\sigma_0 = a$ and that $G_0 \subset F_{\mathbf{v}}^\infty$. Hence we deduce that:

$$\begin{aligned} \text{ord}_{s=a} Y(G; s) &= \text{codim}G_0 \geq \text{codim}F_{\mathbf{v}}^\infty = q - \dim F_{\mathbf{v}}^\infty \\ &\geq q - \text{rank}(\{\boldsymbol{\mu}^i \mid i = 1, \dots, r\}) + 1 = q - \text{rank}(\{\boldsymbol{\alpha}^i \mid i = 1, \dots, q\}) + 1 \\ &\geq q - \text{rank}(I) + 1 = \sum_{\boldsymbol{\alpha} \in I} u(\boldsymbol{\alpha}) - \text{rank}(I) + 1. \end{aligned} \quad (27)$$

Using the relation $\Gamma(v+1) = v\Gamma(v)$ we also see that for $\sigma \gg 1$,

$$\mathcal{R}(s) := \frac{1}{(2\pi i)^r} \int_{\rho_1 - i\infty}^{\rho_1 + i\infty} \dots \int_{\rho_r - i\infty}^{\rho_r + i\infty} \frac{L(s; \mathbf{z}) d\mathbf{z}}{(s - a - \langle \mathbf{1}, \mathbf{a} \rangle) \prod_{\boldsymbol{\alpha} \in I} \langle \boldsymbol{\alpha}, \mathbf{z} \rangle^{u(\boldsymbol{\alpha})}},$$

where $L(s; \mathbf{z}) := \Gamma(s - (a - 1) - z_1 - \dots - z_r) \prod_{i=1}^r \Gamma(a_i + z_i) \Gamma(s)^{-1} \prod_{k=1}^r h_k^{-a_k - z_k}$. Lemma 2 imply that $L(s; \mathbf{z})$ satisfies the assumptions of lemma 3. In particular it imply that $L(s; \mathbf{z})$ satisfies the estimate (11). Therefore it follows from lemma 3 (with $\phi = \mathbf{1}$, $q = 1$ and $\sigma_1 = a$) that $\text{ord}_{s=a} Y(G; s) = \text{ord}_{s=a} \mathcal{R}(s) \leq 1 + \sum_{\boldsymbol{\alpha} \in I} u(\boldsymbol{\alpha}) - \text{rank}(I)$. This and (27) imply that $\text{ord}_{s=a} Y(G; s) = \sum_{\boldsymbol{\alpha} \in I} u(\boldsymbol{\alpha}) - \text{rank}(I) + 1$.

This completes the proof of lemma 4. \diamond

4.2 Proof of the first part of theorem 3

Fix in the sequel of this proof a point $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{R}_+^{*n}$ and a pair of regularization $\mathcal{T}_{\mathbf{c}} = (I_{\mathbf{c}}; \mathbf{u})$ of $\mathcal{M}(f; \cdot)$ at \mathbf{c} as in Definition 2 in §3.2.1.

Since $\sum \frac{|f(m_1, \dots, m_n)|}{(m_1 \dots m_n)^t} < +\infty$ for $t = 1 + \sup_i c_i$, we certainly have

$$f(m_1, \dots, m_n) \ll (m_1 \dots m_n)^t \text{ uniformly in } \mathbf{m} \in \mathbb{N}^n. \quad (28)$$

Let $P := \sum_{k=1}^r b_k \mathbf{X}^{\boldsymbol{\gamma}^k}$ be a generalized polynomial with positive coefficients, elliptic and homogeneous of degree $d > 0$. Since P is elliptic, we have $\text{con}^*(\text{supp}(P)) := \sum_{i=1}^r \mathbb{R}_+^* \boldsymbol{\gamma}^k = \mathbb{R}_+^{*n}$. Therefore

$\mathbf{c} \in \text{con}^*(\text{supp}(P))$ implies there exists $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{R}_+^r$, such that

$$\mathbf{c} = \sum_{k=1}^r a_k \gamma^k. \quad (29)$$

It follows that we have uniformly in $\mathbf{x} \in [1, +\infty[^n$:

$$(x_1^{c_1} \dots x_n^{c_n})^{1/|\mathbf{a}|} = \left(\mathbf{x}^{\sum_{k=1}^r (a_k/|\mathbf{a}|) \gamma^k} \right) \ll \sum_{k=1}^r \mathbf{x}^{\gamma^k} \ll P(\mathbf{x}) \ll (x_1 \dots x_n)^d. \quad (30)$$

It is clear that (28) and (30) imply that the series $Z(f; P; s) := \sum_{\mathbf{m} \in \mathbb{N}^n} \frac{f(m_1, \dots, m_n)}{P(m_1, \dots, m_n)^{s/d}}$ has an abscissa of convergence $\sigma_0 < +\infty$.

Moreover, (30) and Taylor's formula imply that $\forall M \in \mathbb{N}$ we have uniformly in $\mathbf{x} \in [1, +\infty[^n$ and $s \in \mathbb{C}$:

$$\begin{aligned} P(\mathbf{x})^{-s/d} &= (1 + P(\mathbf{x}) - 1)^{-s/d} = (1 + P(\mathbf{x}))^{-s/d} \left(1 - \frac{1}{1 + P(\mathbf{x})} \right)^{-s/d} \\ &= \sum_{k=0}^M (-1)^k \binom{-s/d}{k} (1 + P(\mathbf{x}))^{-(s+dk)/d} + \mathcal{O} \left((1 + |s|^{M+1}) (1 + P(\mathbf{x}))^{-(\Re(s)+dM+d)/d} \right). \end{aligned}$$

It follows that for $M \in \mathbb{N}$ and $\sigma > \sigma_0$:

$$\begin{aligned} Z(f; P; s) &= \sum_{k=0}^M (-1)^k \binom{-s/d}{k} Z(f; 1 + P; s + dk) \\ &\quad + \mathcal{O} \left((1 + |s|^{M+1}) Z(|f; 1 + P; \sigma + dM + d) \right). \end{aligned} \quad (31)$$

Thus, it suffices to prove the assertion of the theorem for $Z(f; 1 + P; s)$.

Let $\alpha^1, \dots, \alpha^n$ be n elements of $\mathbb{R}_+^r \setminus \{\mathbf{0}\}$ defined by: $\alpha_k^i = \gamma_i^k$ for all $i = 1, \dots, n$ and $k = 1, \dots, r$. Since $\mathbf{c} = \sum_{k=1}^r a_k \gamma^k$ we have

$$\forall i = 1, \dots, n \quad \langle \alpha^i, \mathbf{a} \rangle = c_i. \quad (32)$$

By using Mellin's formula (9) and (32) we obtain that for any $\boldsymbol{\rho} \in \mathbb{R}_+^{*n}$ and $\sigma > \sup(\sigma_0, d|\mathbf{a}| + d|\boldsymbol{\rho}|)$:

$$\begin{aligned} (2\pi i)^r Z(f; 1 + P; s) &= (2\pi i)^r \sum_{\mathbf{m} \in \mathbb{N}^n} \frac{f(m_1, \dots, m_n)}{(1 + \sum_{k=1}^r b_k \mathbf{m}^{\gamma^k})^{s/d}} \\ &= \sum_{\mathbf{m} \in \mathbb{N}^n} \int_{a_1 + \rho_1 - i\infty}^{a_1 + \rho_1 + i\infty} \dots \int_{a_r + \rho_r - i\infty}^{a_r + \rho_r + i\infty} \frac{\Gamma(s/d - z_1 - \dots - z_r)}{\Gamma(s/d)} \prod_{i=1}^r \frac{\Gamma(z_i)}{b_i^{z_i}} \frac{f(m_1, \dots, m_n)}{(\prod_{k=1}^r \mathbf{m}^{z_k \gamma^k})} d\mathbf{z} \\ &= \sum_{\mathbf{m} \in \mathbb{N}^n} \int_{a_1 + \rho_1 - i\infty}^{a_1 + \rho_1 + i\infty} \dots \int_{a_r + \rho_r - i\infty}^{a_r + \rho_r + i\infty} \frac{\Gamma(s/d - z_1 - \dots - z_r)}{\Gamma(s/d)} \prod_{i=1}^r \frac{\Gamma(z_i)}{b_i^{z_i}} \frac{f(m_1, \dots, m_n)}{(\prod_{i=1}^n m_i^{\langle \alpha^i, \mathbf{z} \rangle})} d\mathbf{z} \\ &= \sum_{\mathbf{m} \in \mathbb{N}^n} \int_{\rho_1 - i\infty}^{\rho_1 + i\infty} \dots \int_{\rho_r - i\infty}^{\rho_r + i\infty} \frac{\Gamma(s/d - |\mathbf{a}| - \sum_{i=1}^r z_i)}{\Gamma(s/d)} \prod_{i=1}^r \frac{\Gamma(a_i + z_i)}{b_i^{a_i + z_i}} \frac{f(m_1, \dots, m_n)}{\prod_{i=1}^n m_i^{c_i + \langle \alpha^i, \mathbf{z} \rangle}} d\mathbf{z} \end{aligned} \quad (33)$$

But for all $\boldsymbol{\beta} \in \mathcal{I}_{\mathbf{c}}$ we have uniformly in $\mathbf{z} \in \mathbb{C}^r$ verifying $\Re(z_i) = \rho_i \forall i = 1, \dots, r$:

$$\Re \left(\sum_{i=1}^n \beta_i (c_i + \langle \alpha^i, \mathbf{z} \rangle) \right) = \langle \mathbf{c}, \boldsymbol{\beta} \rangle + \sum_{i=1}^n \beta_i \langle \alpha^i, \boldsymbol{\rho} \rangle = 1 + \sum_{i=1}^n \beta_i \langle \alpha^i, \boldsymbol{\rho} \rangle.$$

It follows then (see definition 2) that the series

$$\mathcal{M}(f; c_1 + \langle \boldsymbol{\alpha}^1, \mathbf{z} \rangle, \dots, c_n + \langle \boldsymbol{\alpha}^n, \mathbf{z} \rangle) = \sum_{\mathbf{m} \in \mathbb{N}^n} \frac{f(m_1, \dots, m_n)}{\prod_{i=1}^n m_i^{c_i + \langle \boldsymbol{\alpha}^i, \mathbf{z} \rangle}}$$

converges absolutely and uniformly in $\mathbf{z} \in \mathbb{C}^r$ verifying $\Re(z_i) = \rho_i \forall i = 1, \dots, r$.

This with (33), lemma 2 and lemma 1 imply then that for all $\boldsymbol{\rho} \in \mathbb{R}_+^{*n}$ and for all $\sigma > \sup(\sigma_0, d(|\mathbf{a}| + |\boldsymbol{\rho}|))$,

$$\begin{aligned} Z(f; 1 + P; s) &= \frac{1}{(2\pi i)^r} \int_{\rho_1 - i\infty}^{\rho_1 + i\infty} \dots \int_{\rho_r - i\infty}^{\rho_r + i\infty} \Gamma(s/d - |\mathbf{a}| - z_1 - \dots - z_r) \Gamma(s/d)^{-1} \\ &\quad \times \prod_{i=1}^r \frac{\Gamma(a_i + z_i)}{b_i^{a_i + z_i}} \mathcal{M}(f; c_1 + \langle \boldsymbol{\alpha}^1, \mathbf{z} \rangle, \dots, c_n + \langle \boldsymbol{\alpha}^n, \mathbf{z} \rangle) d\mathbf{z} \end{aligned} \quad (34)$$

We now use the hypothesis that $H(f; \mathcal{T}_{\mathbf{c}}; \mathbf{s}) = \left(\prod_{\boldsymbol{\beta} \in I_{\mathbf{c}}} \langle \boldsymbol{\beta}, \mathbf{s} \rangle^{u(\boldsymbol{\beta})} \right) \mathcal{M}(f; \mathbf{c} + \mathbf{s})$ (see (7) in §3.1) has a holomorphic continuation with moderate growth to $\{\mathbf{s} \in \mathbb{C}^n \mid \forall i \Re(s_i) > -\varepsilon\}$ for some positive ε . It is clear that this implies there exists $\varepsilon_1 > 1$ such that

$$\mathbf{z} \mapsto \mathcal{H}(\mathcal{T}_{\mathbf{c}}; \mathbf{z}) := H(f; \mathcal{T}_{\mathbf{c}}; \langle \boldsymbol{\alpha}^1, \mathbf{z} \rangle, \dots, \langle \boldsymbol{\alpha}^n, \mathbf{z} \rangle) \quad (35)$$

has a holomorphic continuation with moderate growth to $\{\mathbf{z} \in \mathbb{C}^r \mid \forall i \Re(z_i) > -\varepsilon_1\}$.

We then rewrite the integrand factor involving \mathcal{M} as follows. For all $\boldsymbol{\beta} \in I_{\mathbf{c}}$, set $\mu(\boldsymbol{\beta}) := \sum_{i=1}^n \beta_i \boldsymbol{\alpha}^i$, and also define:

$$I_{\mathbf{c}}^* = \{\mu(\boldsymbol{\beta}) \mid \boldsymbol{\beta} \in I_{\mathbf{c}}\}, \text{ and } \mathbf{u}^* = (u^*(\boldsymbol{\eta}))_{\boldsymbol{\eta} \in I_{\mathbf{c}}^*}, \text{ where } u^*(\boldsymbol{\eta}) = \sum_{\{\boldsymbol{\beta} \in I_{\mathbf{c}}; \mu(\boldsymbol{\beta}) = \boldsymbol{\eta}\}} u(\boldsymbol{\beta}) \forall \boldsymbol{\eta} \in I_{\mathbf{c}}^*. \quad (36)$$

We conclude as follows:

$$\begin{aligned} &Z(f; 1 + P; s) \\ &= \frac{1}{(2\pi i)^r} \int_{\rho_1 - i\infty}^{\rho_1 + i\infty} \dots \int_{\rho_r - i\infty}^{\rho_r + i\infty} \frac{\Gamma(s/d - |\mathbf{a}| - z_1 - \dots - z_r)}{\Gamma(s/d)} \prod_{i=1}^r \frac{\Gamma(a_i + z_i)}{b_i^{a_i + z_i}} \frac{\mathcal{H}(\mathcal{T}_{\mathbf{c}}; \mathbf{z})}{\prod_{\boldsymbol{\eta} \in I_{\mathbf{c}}^*} (\langle \boldsymbol{\eta}, \mathbf{z} \rangle)^{u^*(\boldsymbol{\eta})}} d\mathbf{z}. \end{aligned}$$

We deduce from this that for any $\boldsymbol{\rho} \in \mathbb{R}_+^{*r}$ and $\sigma > \sup(\sigma_0, d(|\mathbf{a}| + |\boldsymbol{\rho}|))$:

$$Z(f; 1 + P; s) = \frac{1}{(2\pi i)^r} \int_{\rho_1 - i\infty}^{\rho_1 + i\infty} \dots \int_{\rho_r - i\infty}^{\rho_r + i\infty} \frac{U(s; \mathbf{z}) \mathcal{H}(\mathcal{T}_{\mathbf{c}}; \mathbf{z}) d\mathbf{z}}{(s - d|\mathbf{a}| - \langle d\mathbf{1}, \mathbf{z} \rangle) \prod_{\boldsymbol{\eta} \in I_{\mathbf{c}}^*} (\boldsymbol{\eta}, \mathbf{z})^{u^*(\boldsymbol{\eta})}} \quad (37)$$

where

$$\begin{aligned} U(s; \mathbf{z}) &:= d(s/d - |\mathbf{a}| - \langle \mathbf{1}, \mathbf{z} \rangle) \Gamma(s/d - |\mathbf{a}| - \langle \mathbf{1}, \mathbf{z} \rangle) \Gamma(s/d)^{-1} \left(\prod_{i=1}^r \Gamma(a_i + z_i) b_i^{-a_i - z_i} \right) \\ &= d \Gamma(s/d - |\mathbf{a}| - z_1 - \dots - z_r + 1) \Gamma(s/d)^{-1} \left(\prod_{i=1}^r \Gamma(a_i + z_i) b_i^{-a_i - z_i} \right). \end{aligned} \quad (38)$$

It is well known that the Euler function $z \mapsto \Gamma(z)$ is holomorphic and has no zeros in the half-plane $\{z \in \mathbb{C} \mid \Re(z) > 0\}$. It follows that $(s, \mathbf{z}) \mapsto U(s, \mathbf{z})$ is holomorphic in

$$\mathcal{D}_r(2\delta_1, 2\delta_1) = \{(s, \mathbf{z}) = (\sigma + i\tau, \mathbf{x} + i\mathbf{y}) \in \mathbb{C} \times \mathbb{C}^r \mid \sigma > d|\mathbf{a}| - 2\delta_1 \text{ and } |\Re(z_i)| < 2\delta_1 \forall i = 1, \dots, r\},$$

where $\delta_1 := \frac{1}{4} \inf(d |\mathbf{a}|, \frac{1+|\mathbf{a}|}{r+1/d}, a_1, \dots, a_r) > 0$. Moreover lemma 2 implies that there exists $B_0 = B_0(\mathbf{a}, \mathbf{b}, d, r) > 0$ such that we have uniformly in $(s; \mathbf{z}) \in \mathcal{D}(2\delta_1, 2\delta_1)$ the estimate:

$$U(s; \mathbf{z}) \ll_{\sigma, \mathbf{a}, \mathbf{b}, d, r} (1 + |\tau|)^{2\frac{|\sigma|}{d} + B_0} \prod_{i=1}^r (1 + |y_i|)^{\frac{|\sigma|}{d} + B_0} e^{\frac{\pi}{2} (|\frac{\tau}{d}| - \sum_{i=1}^r |y_i| - |\frac{\tau}{d} - \sum_{i=1}^r y_i|)}. \quad (39)$$

From (35) we also know that there exist $A_1, B_1 > 0$ such that $\mathcal{H}(\mathcal{T}_{\mathbf{c}}; \mathbf{z})$ satisfies the following estimate on $\{\mathbf{z} \in \mathbb{C}^r \mid \forall i \Re(z_i) > -\varepsilon_1\}$:

$$\mathcal{H}(\mathcal{T}_{\mathbf{c}}; \mathbf{z}) \ll_{\Re(\mathbf{z})} (1 + |\Im(\mathbf{z})|)^{A_1 |\Re(\mathbf{z})| + B_1} \ll_{\Re(\mathbf{z})} \prod_{i=1}^r (1 + |\Im(z_i)|)^{A_1 |\Re(\mathbf{z})| + B_1}. \quad (40)$$

Let $\delta := \inf(\varepsilon_1/2, \delta_1)$. Putting together the two preceding estimates (39) and (40)) we conclude that there exists $B > 0$ such that :

$V(s, z) := U(s, z) \mathcal{H}(\mathcal{T}_{\mathbf{c}}; \mathbf{z})$ is holomorphic in $\mathcal{D}_r(2\delta, 2\delta)$ and satisfies in it the estimate:

$$V(s; \mathbf{z}) \ll_{\sigma} (1 + |\tau|)^{2\frac{|\sigma|}{d} + (|\mathbf{a}| + B)} \prod_{i=1}^r (1 + |y_i|)^{2\frac{|\sigma|}{d} + B} \times e^{\frac{\pi}{2} (|\frac{\tau}{d}| - \sum_{i=1}^r |y_i| - |\frac{\tau}{d} - \sum_{i=1}^r y_i|)}. \quad (41)$$

We can therefore apply Lemma 3 to (37) by setting

$$L(s, \mathbf{z}) = V(s, \mathbf{z}), \quad \sigma_1 = d |\mathbf{a}|, \quad \phi = d \mathbf{1} \quad \text{and} \quad q = 1.$$

We conclude that there exists $\eta > 0$ such that $s \mapsto Z(f; 1 + P; s)$ has a meromorphic continuation with moderate growth to the half-plane $\{\sigma > d|\mathbf{a}| - \eta\}$ with at most one pole at $s = d|\mathbf{a}|$ of order at most $\rho_0^* := \sum_{\boldsymbol{\eta} \in I_{\mathbf{c}}^*} u^*(\boldsymbol{\eta}) - \text{rank}(I_{\mathbf{c}}^*) + 1$.

Moreover it follows from the ellipticity of the polynomial P that $\text{rank}(\{\boldsymbol{\alpha}^1, \dots, \boldsymbol{\alpha}^n\}) = \text{rank}(\{\boldsymbol{\gamma}^1, \dots, \boldsymbol{\gamma}^r\}) = \text{rank}(\text{supp}(P)) = n$. This implies that $\text{rank}(I_{\mathbf{c}}^*) = \text{rank}(I_{\mathbf{c}})$. We conclude:

$$\rho_0^* := \sum_{\boldsymbol{\eta} \in I_{\mathbf{c}}^*} u^*(\boldsymbol{\eta}) - \text{rank}(I_{\mathbf{c}}^*) + 1 = \sum_{\boldsymbol{\beta} \in I_{\mathbf{c}}} u(\boldsymbol{\beta}) - \text{rank}(I_{\mathbf{c}}) + 1. \quad (42)$$

Since

$$|\mathbf{c}| = \langle \mathbf{1}, \mathbf{c} \rangle = \langle \mathbf{1}, \sum_{k=1}^r a_k \boldsymbol{\gamma}^k \rangle = \sum_{k=1}^r a_k |\boldsymbol{\gamma}^k| = \sum_{k=1}^r a_k d = d|\mathbf{a}|, \quad (43)$$

the proof of the first part of Theorem 3 now follows. \diamond

4.3 Proof of the second part of Theorem 3

Notations used in §4.2 will also be used throughout this section.

We assume in addition that the point $\mathbf{c} \in \mathbb{R}_+^{*n}$ and the pair of regularization $\mathcal{T}_{\mathbf{c}} = (I_{\mathbf{c}}; \mathbf{u})$ satisfy the two following assumptions:

1. $\mathbf{1} \in \text{con}^*(I_{\mathbf{c}})$;
2. there exists a function K (holomorphic in a tubular neighborhood of $\mathbf{0}$) such that:
 $H(f; \mathcal{T}_{\mathbf{c}}; \mathbf{s}) = K(\langle \boldsymbol{\beta}, \mathbf{s} \rangle_{\boldsymbol{\beta} \in I_{\mathbf{c}}})$.

Recall also from (43) that $|\mathbf{c}| = d|\mathbf{a}|$.

From (31) we conclude that the proof of Theorem 3 will follow once we prove that $|\mathbf{c}|$ is a pole of $Z(f; 1 + P; s)$ of order at most ρ_0^* (see (42)) and

$$Z(f; 1 + P; s) = \frac{H(f; \mathcal{T}_{\mathbf{c}}; \mathbf{0}) d^{\rho_0^*} A_0(\mathcal{T}_{\mathbf{c}}; P)}{(s - |\mathbf{c}|)^{\rho_0^*}} + O\left(\frac{1}{(s - |\mathbf{c}|)^{\rho_0^* - 1}}\right) \text{ as } s \rightarrow |\mathbf{c}|.$$

We will prove this in the sequel. Our strategy is the following:

First we set $\tilde{H}(\mathbf{z}) := \mathcal{H}(\mathcal{T}_{\mathbf{c}}; \mathbf{z}) - \mathcal{H}(\mathcal{T}_{\mathbf{c}}; \mathbf{0})$, where $\mathcal{H}(\mathcal{T}_{\mathbf{c}}; \mathbf{z})$ is defined by (35). Thus, $\mathcal{H}(\mathcal{T}_{\mathbf{c}}; \mathbf{0}) = H(f; \mathcal{T}_{\mathbf{c}}; \mathbf{0})$.

Let $U(s; \mathbf{z})$ be the function defined in (38). It follows that (37) can then be written in this way:

$$\forall \rho \in \mathbb{R}_+^{*r} \text{ and } \sigma > \sup(\sigma_0, d(|\mathbf{a}| + |\rho|)), \quad Z(f; 1 + P; s) = H(f; \mathcal{T}_{\mathbf{c}}; \mathbf{0}) Z_1(s) + Z_2(s), \quad (44)$$

where

$$\begin{aligned} Z_1(s) &= \frac{1}{(2\pi i)^r} \int_{\rho_1 - i\infty}^{\rho_1 + i\infty} \cdots \int_{\rho_r - i\infty}^{\rho_r + i\infty} \frac{U(s; \mathbf{z}) d\mathbf{z}}{(s - d|\mathbf{a}| - \langle d\mathbf{1}, \mathbf{z} \rangle) \prod_{\eta \in I_{\mathbf{c}}^*} (\langle \eta, \mathbf{z} \rangle)^{u^*(\eta)}} \\ Z_2(s) &= \frac{1}{(2\pi i)^r} \int_{\rho_1 - i\infty}^{\rho_1 + i\infty} \cdots \int_{\rho_r - i\infty}^{\rho_r + i\infty} \frac{U(s; \mathbf{z}) \tilde{H}(\mathbf{z}) d\mathbf{z}}{(s - d|\mathbf{a}| - \langle d\mathbf{1}, \mathbf{z} \rangle) \prod_{\eta \in I_{\mathbf{c}}^*} (\langle \eta, \mathbf{z} \rangle)^{u^*(\eta)}}. \end{aligned}$$

In section §4.3.1, we will use Lemma 4 to prove that $|\mathbf{c}| = d|\mathbf{a}|$ is a pole of $Z_1(s)$ of order ρ_0^* and even to determine the top order term in the principal part of $Z_1(s)$ at $|\mathbf{c}|$.

The integral $Z_2(s)$ is more complicated and there is no hope to get for it a precise result like for $Z_1(s)$. Moreover if we can only infer that $Z_2(s)$ has a possible pole at $s = |\mathbf{c}|$ of order at most ρ_0^* . As a result, we would not yet be able to prove that $s = |\mathbf{c}|$ is a pole of $Z(f; 1 + P; s)$! To get around this difficulty, we will use in §4.3.2 the crucial Lemma 3, which give a very precise estimate of the order of the possible pole $s = |\mathbf{c}|$ since it implies that $Z_2(s)$ has a pole at $s = |\mathbf{c}|$ of order at most $\rho_0^* - 1$. Combining this with the result on $Z_1(s)$ suffices to complete the proof of Theorem 3.

4.3.1 The principal part of $Z_1(s)$ at its first pole

It is easy to see that for $\sigma \gg 1$:

$$Z_1(s) = \frac{1}{(2\pi i)^r} \int_{\rho_1 - i\infty}^{\rho_1 + i\infty} \cdots \int_{\rho_r - i\infty}^{\rho_r + i\infty} \frac{\Gamma(s/d - |\mathbf{a}| - z_1 - \cdots - z_r) \prod_{k=1}^r \Gamma(a_k + z_k) d\mathbf{z}}{\Gamma(s/d) \prod_{k=1}^r b_k^{a_k + z_k} \prod_{\eta \in I_{\mathbf{c}}^*} (\langle \eta, \mathbf{z} \rangle)^{u^*(\eta)}}.$$

By our hypotheses, we know that $\mathbf{1} \in \text{con}(I_{\mathbf{c}}^*)$, and $\langle \eta, \mathbf{a} \rangle = \sum_{i=1}^n \beta_i \langle \alpha^i, \mathbf{a} \rangle = \sum_{i=1}^n \beta_i c_i = \langle \beta, \mathbf{c} \rangle = 1$ $\forall \eta = \mu(\beta) \in I_{\mathbf{c}}^*$. Thus, it follows from lemma 4 that $s = d|\mathbf{a}| = |\mathbf{c}|$ is a pole of $Z_1(s)$ of order ρ_0^* (see (42)) and that

$$Z_1(s) \underset{s \rightarrow |\mathbf{c}|}{\sim} \frac{d^{\rho_0^*} A_0(I_{\mathbf{c}}^*; \mathbf{u}^*; \mathbf{b})}{(s - |\mathbf{c}|)^{\rho_0^*}} \quad (45)$$

where $A_0(I_{\mathbf{c}}^*; \mathbf{u}^*; \mathbf{b}) > 0$ is the *volume constant* associated to $I_{\mathbf{c}}^*$, \mathbf{u}^* and \mathbf{b} (see §2.3.2).

4.3.2 A sharper estimate for $\text{ord}_{s=|\mathbf{c}|} Z_2(s)$ and end of the proof of Theorem 3

The quantities σ_1, ϕ , and q , introduced during the proof of lemma 3 are assigned values here by setting: $\sigma_1 := |\mathbf{c}|$, $\phi := d\mathbf{1} = (d, \dots, d) \in \mathbb{R}^r$ and $q := 1$.

We also define $L(s; \mathbf{z}) := U(s; \mathbf{z}) \tilde{H}(\mathbf{z})$ where $\tilde{H}(\mathbf{z}) := \mathcal{H}(\mathcal{T}_{\mathbf{c}}; \mathbf{z}) - \mathcal{H}(\mathcal{T}_{\mathbf{c}}; \mathbf{0})$ as above.

As a result, we see that $Z_2(s) = T_r(s)$ (see (12) for the definition of $T_r(s)$).

It also follows that the role played by the set I in §4.1.2 is played here by the set $I_{\mathbf{c}}^*$ and the exponents $u(\boldsymbol{\alpha})$ become the $u^*(\boldsymbol{\eta})$ (see (36)). Thus, the quantity d_r from lemma 3 is as follows (see (42)):

$$d_r = \sum_{\boldsymbol{\eta} \in I_{\mathbf{c}}^*} u^*(\boldsymbol{\eta}) - \text{rank}(I_{\mathbf{c}}^*) + 1 - \varepsilon_0(\boldsymbol{\phi}, L) = \rho_0^* - \varepsilon_0(\boldsymbol{\phi}, L).$$

The estimates (39) and (40) imply that there exists $\delta > 0$ such that $L(s; \mathbf{z})$ is holomorphic in $\mathcal{D}_r(2\delta, 2\delta)$, on which the estimate (11) is satisfied (with a suitable choice of $A, B > 0$). Lemma 3 implies then that there exists $\eta > 0$ such that $s \mapsto Z_2(s)$ has a meromorphic continuation to the half-plane $\{\sigma > \sigma_1 - \eta\}$ with only one possible pole at $s = \sigma_1 = |\mathbf{c}|$. Lemma 3 implies also the *crucial* estimate:

$$\text{ord}_{s=|\mathbf{c}|} Z_2(s) \leq \rho_0^* - \varepsilon_0(\boldsymbol{\phi}, L). \quad (46)$$

Thus, we must evaluate $\varepsilon_0(\boldsymbol{\phi}, L)$. We do this as follows.

Since $\mathbf{1} \in \text{con}^*(I_{\mathbf{c}})$, there exists a set $\{t_{\boldsymbol{\beta}}\}_{\boldsymbol{\beta} \in I_{\mathbf{c}}} \subset \mathbb{R}_+^*$ such that

$\mathbf{1} = \sum_{\boldsymbol{\beta} \in I_{\mathbf{c}}} t_{\boldsymbol{\beta}} \boldsymbol{\beta}$ (i.e. $\sum_{\boldsymbol{\beta} \in I_{\mathbf{c}}} t_{\boldsymbol{\beta}} \beta_i = 1 \ \forall i = 1, \dots, n$). Consequently we have:

$$\sum_{\boldsymbol{\beta} \in I_{\mathbf{c}}} t_{\boldsymbol{\beta}} \mu(\boldsymbol{\beta}) = \sum_{\boldsymbol{\beta} \in I_{\mathbf{c}}} t_{\boldsymbol{\beta}} \sum_{i=1}^n \beta_i \boldsymbol{\alpha}^i = \sum_{i=1}^n \left(\sum_{\boldsymbol{\beta} \in I_{\mathbf{c}}} t_{\boldsymbol{\beta}} \beta_i \right) \boldsymbol{\alpha}^i = \sum_{i=1}^n \boldsymbol{\alpha}^i = \sum_{k=1}^r \left(\sum_{i=1}^n \gamma_i^k \right) \mathbf{e}_k = d \mathbf{1}.$$

We conclude from this that

$$\boldsymbol{\phi} := d \mathbf{1} \in \text{con}^*(I_{\mathbf{c}}^*) \setminus \{\mathbf{0}\}. \quad (47)$$

Furthermore, assumption 2 implies that there exists a function K (holomorphic in a tubular neighborhood of $\mathbf{0}$) such that $H(f; \mathcal{T}_{\mathbf{c}}, \mathbf{s}) = K(\langle \boldsymbol{\beta}, \mathbf{s} \rangle_{\boldsymbol{\beta} \in I_{\mathbf{c}}})$. But for any $\boldsymbol{\beta} \in I_{\mathbf{c}}$ and for any $\mathbf{z} \in \mathbb{C}^r$ we have,

$$\sum_{i=1}^n \beta_i \langle \boldsymbol{\alpha}^i, \mathbf{z} \rangle = \left\langle \sum_{i=1}^n \beta_i \boldsymbol{\alpha}^i, \mathbf{z} \right\rangle = \langle \mu(\boldsymbol{\beta}), \mathbf{z} \rangle.$$

It follows that: $\mathcal{H}(\mathcal{T}_{\mathbf{c}}; \mathbf{z}) := H(f; \mathcal{T}_{\mathbf{c}}; \langle \boldsymbol{\alpha}^1, \mathbf{z} \rangle, \dots, \langle \boldsymbol{\alpha}^n, \mathbf{z} \rangle) = K(\langle \mu(\boldsymbol{\beta}), \mathbf{z} \rangle_{\boldsymbol{\beta} \in I_{\mathbf{c}}})$. Consequently there exists a function \tilde{K} (holomorphic in a tubular neighborhood of $\mathbf{0}$) such that: $\tilde{H}(\mathbf{z}) = \mathcal{H}(\mathcal{T}_{\mathbf{c}}; \mathbf{z}) - \mathcal{H}(\mathcal{T}_{\mathbf{c}}; \mathbf{0}) = \tilde{K}(\langle \boldsymbol{\eta}, \mathbf{z} \rangle_{\boldsymbol{\eta} \in I_{\mathbf{c}}^*})$. Since in addition we have $\tilde{H}(\mathbf{0}) = 0$ and $\boldsymbol{\phi} = d \mathbf{1} \in \text{con}^*(I_{\mathbf{c}}^*) \setminus \{\mathbf{0}\}$, it follows from lemma 3 that $\varepsilon_0(\boldsymbol{\phi}, L) = 1$. Thus, we conclude from this and (46) that

$$\text{ord}_{s=|\mathbf{c}|} Z_2(s) \leq \rho_0^* - 1.$$

Combining this with (45), (44), and (31) implies that

$$Z(f; P; s) = \frac{H(f; \mathcal{T}_{\mathbf{c}}; \mathbf{0}) d^{\rho_0^*} A_0(I_{\mathbf{c}}^*; \mathbf{u}^*; \mathbf{b})}{(s - |\mathbf{c}|)^{\rho_0^*}} + O\left(\frac{1}{(s - |\mathbf{c}|)^{\rho_0^* - 1}}\right) \text{ as } s \rightarrow |\mathbf{c}|.$$

This completes the proof of theorem 3, once one has also noted that $A_0(I_{\mathbf{c}}^*; \mathbf{u}^*; \mathbf{b}) = A_0(\mathcal{T}_{\mathbf{c}}, P)$ where $A_0(\mathcal{T}_{\mathbf{c}}, P)$ is the mixed volume constant associated to P and $\mathcal{T}_{\mathbf{c}}$ as in §2.3.3. \diamond

5 Proofs of Theorem 1 and 2

5.1 A Lemma from convex analysis and its proof

Using the definitions introduced in §2.2, I will first give a lemma from convex analysis.

Lemma 5. *Let I be a nonempty subset of $\mathbb{R}_+^n \setminus \{\mathbf{0}\}$. Set $\mathcal{E}(I)$ to be its Newton polyhedron and denotes by $\mathcal{E}^\circ(I)$ its dual (see §2.2). Let F be a face of $\mathcal{E}(I)$ that is not contained in a coordinate hyperplane and $\mathbf{c} \in \text{pol}_0(F)$ a normalized polar vector of F . Then: F meets the diagonal if and only if $|\mathbf{c}| = \iota(I)$ where $\iota(I) = \min\{|\boldsymbol{\alpha}|; \boldsymbol{\alpha} \in \mathcal{E}^\circ(I) \cap \mathbb{R}_+^n\}$.*

Proof of Lemma 5

We first note that the definition of the normalized polar vector implies that $\mathbf{c} \in \mathcal{E}(I)^\circ \cap \mathbb{R}_+^n$.

• Assume first that the diagonal Δ meets the face F of the Newton Polyhedron $\mathcal{E}(I)$. Therefore, there exists $t_0 > 0$ such that $\Delta \cap F = \{t_0 \mathbf{1}\}$. Let $\boldsymbol{\alpha}^1, \dots, \boldsymbol{\alpha}^r \in I \cap F$ and let J a subset (possibly empty) of $\{1, \dots, n\}$ such that $F = \text{convex hull}\{\boldsymbol{\alpha}^1, \dots, \boldsymbol{\alpha}^r\} + \text{con}\{\mathbf{e}_i \mid i \in J\}$. Thus there exist $\lambda_1, \dots, \lambda_r \in \mathbb{R}_+$ verifying $\lambda_1 + \dots + \lambda_r = 1$ and a finite family $(\mu_i)_{i \in J}$ of elements of \mathbb{R}_+ such that:

$$t_0 \mathbf{1} = \sum_{i=1}^r \lambda_i \boldsymbol{\alpha}^i + \sum_{j \in J} \mu_j \mathbf{e}_j. \quad (48)$$

But \mathbf{c} is orthogonal to the vectors \mathbf{e}_j ($j \in J$) and $\langle \mathbf{c}, \boldsymbol{\alpha}^i \rangle = 1$ for all $i = 1, \dots, r$. Thus it follows from the relation (48) that $t_0 |\mathbf{c}| = \langle \mathbf{c}, t_0 \mathbf{1} \rangle = \sum_{i=1}^r \lambda_i \langle \boldsymbol{\alpha}^i, \mathbf{c} \rangle + \sum_{j \in J} \mu_j \langle \mathbf{e}_j, \mathbf{c} \rangle = \sum_{i=1}^r \lambda_i = 1$. Consequently $|\mathbf{c}| = t_0^{-1}$.

Relation (48) implies also that for all $\mathbf{b} \in \mathcal{E}(I)^\circ \cap \mathbb{R}_+^n$:

$$|\mathbf{b}| = t_0^{-1} \langle \mathbf{b}, t_0 \mathbf{1} \rangle = t_0^{-1} \sum_{i=1}^r \lambda_i \langle \boldsymbol{\alpha}^i, \mathbf{b} \rangle + \sum_{j \in J} \mu_j b_j \geq t_0^{-1} \sum_{i=1}^r \lambda_i \langle \boldsymbol{\alpha}^i, \mathbf{b} \rangle \geq t_0^{-1} \sum_{i=1}^r \lambda_i = t_0^{-1} = |\mathbf{c}|.$$

This implies $|\mathbf{c}| = \iota(I)$.

• Conversely assume that $|\mathbf{c}| = \iota(I)$. We will show that $\Delta \cap F \neq \emptyset$.

Let G be a face of $\mathcal{E}(I)$ which meets the diagonal Δ . Since G is not included in the coordinate hyperplanes, it has a normalized polar vector $\mathbf{a} \in \text{pol}_0(G)$. Moreover there exist $\boldsymbol{\beta}^1, \dots, \boldsymbol{\beta}^k \in I \cap G$ and T a subset (possibly empty) of $\{1, \dots, n\}$ such that $G = \text{convex hull}\{\boldsymbol{\beta}^1, \dots, \boldsymbol{\beta}^k\} + \text{con}\{\mathbf{e}_i \mid i \in T\}$.

The proof of the first part shows then that $|\mathbf{a}| = \iota(I)$ and that there exist $\nu_1, \dots, \nu_k \in \mathbb{R}_+$ verifying $\nu_1 + \dots + \nu_k = 1$ and a finite family $(\delta_j)_{j \in T}$ of elements of \mathbb{R}_+ such that if we set $t_0 := |\mathbf{a}|^{-1}$, then:

$$t_0 \mathbf{1} = \sum_{i=1}^k \nu_i \boldsymbol{\beta}^i + \sum_{j \in T} \delta_j \mathbf{e}_j \in G \cap \Delta. \quad (49)$$

Set $T' := \{j \in T \mid \delta_j \neq 0\}$. It follows from relation (49) that

$$|\mathbf{c}| = t_0^{-1} \langle \mathbf{c}, t_0 \mathbf{1} \rangle = t_0^{-1} \sum_{i=1}^k \nu_i \langle \boldsymbol{\beta}^i, \mathbf{c} \rangle + \sum_{j \in T'} \delta_j \langle \mathbf{e}_j, \mathbf{c} \rangle \geq t_0^{-1} \sum_{i=1}^k \nu_i \langle \boldsymbol{\beta}^i, \mathbf{c} \rangle \geq t_0^{-1} \sum_{i=1}^k \nu_i = t_0^{-1} = |\mathbf{a}|.$$

But $|\mathbf{a}| = \iota(I) = |\mathbf{c}|$ so the intermediate inequalities must be equalities. This clearly forces $\langle \boldsymbol{\beta}^i, \mathbf{c} \rangle = 1 \forall i = 1, \dots, k$ and $\langle \mathbf{c}, \mathbf{e}_j \rangle = 0 \forall j \in T'$. Relation (49) implies then that:

$$\langle t_0 \mathbf{1}, \mathbf{c} \rangle = \sum_{i=1}^k \nu_i \langle \boldsymbol{\beta}^i, \mathbf{c} \rangle + \sum_{j \in T'} \delta_j \langle \mathbf{e}_j, \mathbf{c} \rangle = \sum_{i=1}^k \nu_i = 1.$$

Since $t_0 \mathbf{1} \in \mathcal{E}(I)$, it follows from the preceding discussion that $t_0 \mathbf{1} \in F$ and therefore $\Delta \cap F \neq \emptyset$. This finishes the proof of Lemma 5. \diamond

5.2 Proof of Proposition 2

For all $\varepsilon \in \mathbb{R}$, set $U_\varepsilon := \{\mathbf{s} \in \mathbb{C}^n \mid \Re(s_i) > \varepsilon \forall i = 1, \dots, n\}$.

• Let $\mathbf{c} \in \text{pol}_0(\mathcal{F}_0(f))$. The compactness of the face $\mathcal{F}_0(f)$ implies that $\mathbf{c} \in \mathbb{R}_+^n$. Moreover it

follows from the definition of $pol_0(\mathcal{F}_0(f))$ that $\forall \beta \in S^*(g) \langle \beta, \mathbf{c} \rangle \geq 1$ with equality if and only if $\beta \in \mathcal{F}_0(f) \cap S^*(g)$.

Set $\delta_0 := \frac{1}{2} \inf_{i=1, \dots, n} c_i > 0$ and $\Omega_{\mathbf{c}} := \{\mathbf{s} \in \mathbb{C}^n \mid \forall i \Re(s_i) > c_i\}$.

Fix also $N := \left\lceil \frac{8}{\delta_0} + \sup_{\mathbf{x} \in \mathcal{F}_0(f)} |\mathbf{x}| \right\rceil + 1 \in \mathbb{N}$. (Evidently, $N < \infty$ since $\mathcal{F}_0(f)$ is compact.)

It is easy to see that the following bound is uniform in p and $\mathbf{s} \in U_{-\delta_0} = \{\mathbf{s} \in \mathbb{C}^n \mid \forall i \Re(s_i) > -\delta_0\}$:

$$\sum_{|\nu| \geq N+1} \frac{g(\nu)}{p^{\langle \mathbf{c} + \mathbf{s}, \nu \rangle}} \ll \sum_{|\nu| \geq N+1} \frac{|\nu|^D}{p^{\delta_0 |\nu|}} \ll \frac{1}{p^{\delta_0(N+1)/2}} \sum_{|\nu| \geq N+1} \frac{|\nu|^D}{2^{\delta_0 |\nu|/2}} \ll \frac{1}{p^{\delta_0(N+1)/2}} \ll \frac{1}{p^2}.$$

Thus, the following is uniform in p and $\mathbf{s} \in U_{-\delta_0}$:

$$\begin{aligned} \sum_{|\nu| \geq 1} \frac{f(p^{\nu_1}, \dots, p^{\nu_n})}{p^{\langle \mathbf{c} + \mathbf{s}, \nu \rangle}} &= \sum_{|\nu| \geq 1} \frac{g(\nu)}{p^{\langle \mathbf{c} + \mathbf{s}, \nu \rangle}} = \sum_{1 \leq |\nu| \leq N} \frac{g(\nu)}{p^{\langle \mathbf{c} + \mathbf{s}, \nu \rangle}} + O\left(\frac{1}{p^2}\right) \\ &= \sum_{\nu \in \mathcal{F}_0(f) \cap S^*(g)} \frac{g(\nu)}{p^{1 + \langle \mathbf{s}, \nu \rangle}} + \sum_{\substack{1 \leq |\nu| \leq N \\ \nu \notin \mathcal{F}_0(f)}} \frac{g(\nu)}{p^{\langle \mathbf{c}, \nu \rangle + \langle \mathbf{s}, \nu \rangle}} + O\left(\frac{1}{p^2}\right) \end{aligned} \quad (50)$$

Since $\mathcal{F}_0(f) \cap S^*(g)$ is a finite set and $\langle \mathbf{c}, \nu \rangle > 1$ for all $\nu \in S^*(g) \setminus \mathcal{F}_0(f)$, it follows from (50) that there exists $\delta_1 \in]0, \delta_0[$ and $\varepsilon_1 > 0$ such that the following is uniform in p and $\mathbf{s} \in U_{-\delta_1}$:

$$\sum_{|\nu| \geq 1} \frac{f(p^{\nu_1}, \dots, p^{\nu_n})}{p^{\langle \mathbf{c} + \mathbf{s}, \nu \rangle}} = \sum_{|\nu| \geq 1} \frac{g(\nu)}{p^{\langle \mathbf{c} + \mathbf{s}, \nu \rangle}} = \sum_{\nu \in \mathcal{F}_0(f) \cap S^*(g)} \frac{g(\nu)}{p^{1 + \langle \mathbf{s}, \nu \rangle}} + O\left(\frac{1}{p^{1 + \varepsilon_1}}\right) \quad (51)$$

The multiplicativity of f now implies that $\mathcal{M}(f; \mathbf{s})$ converges absolutely in $\Omega_{\mathbf{c}} := \{\mathbf{s} \in \mathbb{C}^n \mid \forall i \Re(s_i) > c_i\}$ on which it can be written as follows:

$$\mathcal{M}(f; \mathbf{s}) := \sum_{\mathbf{m} \in \mathbb{N}^n} \frac{f(m_1, \dots, m_n)}{m^{s_1} \dots m^{s_n}} = \prod_p \left(\sum_{\nu \in \mathbb{N}_0^n} \frac{f(p^{\nu_1}, \dots, p^{\nu_n})}{p^{\langle \nu, \mathbf{s} \rangle}} \right) = \prod_p \left(\sum_{\nu \in \mathbb{N}_0^n} \frac{g(\nu)}{p^{\langle \nu, \mathbf{s} \rangle}} \right) \quad (52)$$

We conclude that for all $\mathbf{s} \in U_0 = \{\mathbf{s} \in \mathbb{C}^n \mid \forall i \Re(s_i) > 0\}$:

$$\begin{aligned} G(f; \mathbf{s}) &:= \left(\prod_{\nu \in \mathcal{F}_0(f) \cap S^*(g)} \zeta(1 + \langle \nu, \mathbf{s} \rangle)^{-g(\nu)} \right) \mathcal{M}(f; \mathbf{c} + \mathbf{s}) \\ &= \prod_p \left(\prod_{\nu \in \mathcal{F}_0(f) \cap S^*(g)} \left(1 - \frac{1}{p^{1 + \langle \nu, \mathbf{s} \rangle}} \right)^{g(\nu)} \right) \left(\sum_{\nu \in \mathbb{N}_0^n} \frac{g(\nu)}{p^{\langle \nu, \mathbf{c} + \mathbf{s} \rangle}} \right). \end{aligned} \quad (53)$$

Combining (51) with (53) now implies that there exist $\delta_2 \in]0, \delta_1[$ and $\varepsilon_2 > 0$ such that :

$G(f; \mathbf{s}) = \prod_p \left(1 + O\left(\frac{1}{p^{1 + \varepsilon_2}}\right) \right)$ uniformly in $\mathbf{s} \in U_{-\delta_2}$. It follows that the Euler product $\mathbf{s} \mapsto G(f; \mathbf{s})$ converges absolutely and defines a bounded holomorphic function on $U_{-\delta_2}$.

Moreover, since $G(f; \mathbf{0}) = \prod_p \left(1 - \frac{1}{p} \right)^{\sum_{\nu \in \mathcal{F}_0(f) \cap S^*(g)} g(\nu)} \cdot \left(\sum_{\nu \in \mathbb{N}_0^n} \frac{g(\nu)}{p^{\langle \nu, \mathbf{c} \rangle}} \right)$ is a convergent infinite product whose general term is > 0 , we conclude that $G(f; \mathbf{0}) > 0$.

Moreover, for all $\mathbf{s} \in U_0 = \{\mathbf{s} \in \mathbb{C}^n \mid \forall i \Re(s_i) > 0\}$ we have:

$$\begin{aligned} H(f; \mathcal{T}_{\mathbf{c}}; \mathbf{s}) &:= \left(\prod_{\nu \in \mathcal{F}_0(f) \cap S^*(g)} \langle \nu, \mathbf{s} \rangle^{g(\nu)} \right) \mathcal{M}(f; \mathbf{c} + \mathbf{s}) \\ &= \left(\prod_{\nu \in \mathcal{F}_0(f) \cap S^*(g)} (\langle \nu, \mathbf{s} \rangle \zeta(1 + \langle \nu, \mathbf{s} \rangle))^{g(\nu)} \right) G(f; \mathbf{s}). \end{aligned} \quad (54)$$

Thus, by using the properties of the function $\mathbf{s} \mapsto G(f; \mathbf{s})$ established above and standard properties satisfied by the Riemann zeta function, it follows that there exists $\varepsilon_0 > 0$ such that $\mathbf{s} \mapsto H(f; \mathcal{T}_{\mathbf{c}}; \mathbf{s})$ has a holomorphic continuation with moderate growth to $\{\mathbf{s} \in \mathbb{C}^n \mid \sigma_i > -\varepsilon_0 \forall i\}$.

By (54) we conclude that $H(f; \mathcal{T}_{\mathbf{c}}; \mathbf{0}) = G(f; \mathbf{0}) > 0$.

Moreover, by definition, $\mathcal{F}_0(f)$ is the smallest face of $\mathcal{E}(f)$ which meets the diagonal $\Delta = \mathbb{R}_+ \mathbf{1}$ and $\mathcal{F}_0(f) = \text{conv}(\mathcal{F}_0(f) \cap S^*(g))$. It follows that $\mathbf{1} \in \text{con}^*(\mathcal{F}_0(f) \cap S^*(g))$. This completes the proof of the first part of Proposition 2. \diamond

• We assume now that $\dim \mathcal{F}_0(f) = \text{rank}(S^*(g)) - 1$. Let $I_f = \mathcal{F}_0(f) \cap S^*(g)$.

Set $r := \text{rank}(I_f)$ and fix in the sequel $\boldsymbol{\nu}^1, \dots, \boldsymbol{\nu}^r \in I_f$ such that $\text{rank}\{\boldsymbol{\nu}^1, \dots, \boldsymbol{\nu}^r\} = r$. Set also for any $\varepsilon > 0$ $\mathcal{R}(\varepsilon) := \{\mathbf{z} = (z_1, \dots, z_r) \in \mathbb{C}^r \mid |\Re(z_i)| < \varepsilon \forall i\}$.

The proof of Proposition 2 will be complete once we prove that $\exists \varepsilon > 0$ and a function L such that:

$$L \text{ is holomorphic in } \mathcal{R}(\varepsilon) \text{ and } H(f; \mathcal{T}_{\mathbf{c}}; \mathbf{s}) = L(\langle \boldsymbol{\nu}^1, \mathbf{s} \rangle, \dots, \langle \boldsymbol{\nu}^r, \mathbf{s} \rangle). \quad (55)$$

Since we assume that $\dim \mathcal{F}_0(f) = \text{rank}(S^*(g)) - 1$, it follows that

$$\text{rank}(I_f) = \dim \mathcal{F}_0(f) + 1 = \text{rank}(S^*(g)).$$

By using relations (54) and (53), it is clear that it suffices to prove that (55) holds for the function $G(f; \mathbf{s})$ (instead of $H(f; \mathcal{T}_{\mathbf{c}}; \mathbf{s})$), where

$$G(f; \mathbf{s}) = \prod_p \left(\prod_{\boldsymbol{\nu} \in I_{\mathbf{c}}} \left(1 - \frac{1}{p^{1 + \langle \boldsymbol{\nu}, \mathbf{s} \rangle}} \right)^{g(\boldsymbol{\nu})} \right) \left(\sum_{\boldsymbol{\nu} \in \mathbb{N}_0^n} \frac{g(\boldsymbol{\nu})}{p^{\langle \boldsymbol{\nu}, \mathbf{c} \rangle + \langle \boldsymbol{\nu}, \mathbf{s} \rangle}} \right).$$

First we recall from the proof of Proposition 2 that there exists $\delta_2 > 0$ such that the Euler product $G(f; \mathbf{s})$ converges absolutely and defines a holomorphic function on

$$U_{-\delta_2} := \{\mathbf{s} \in \mathbb{C}^n \mid \Re(s_i) > -\delta_2 \forall i = 1, \dots, n\}.$$

We fix this δ_2 in the following discussion.

It follows from the equality $\text{rank}\{\boldsymbol{\nu}^1, \dots, \boldsymbol{\nu}^r\} = r$ that the linear function $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}^r$, $\mathbf{s} \mapsto \varphi(\mathbf{s}) = (\langle \boldsymbol{\nu}^1, \mathbf{s} \rangle, \dots, \langle \boldsymbol{\nu}^r, \mathbf{s} \rangle)$ is onto. By a permutation of coordinates, if needed, we can then assume that the function $\psi : \mathbb{C}^r \rightarrow \mathbb{C}^r$, $\mathbf{s}' = (s_1, \dots, s_r) \mapsto \psi(\mathbf{s}') := \varphi(\mathbf{s}, \mathbf{0}) = (\sum_{i=1}^r \nu_i^1 s_i, \dots, \sum_{i=1}^r \nu_i^r s_i)$ is an isomorphism. In particular there exist $\boldsymbol{\beta}^1, \dots, \boldsymbol{\beta}^r \in \mathbb{Q}^r$ so that $\forall \mathbf{z} \in \mathbb{C}^r$, $\psi^{-1}(\mathbf{z}) = (\langle \boldsymbol{\beta}^1, \mathbf{z} \rangle, \dots, \langle \boldsymbol{\beta}^r, \mathbf{z} \rangle)$, and

$$\psi^{-1}(\mathcal{R}(\varepsilon_0)) \subset \mathcal{R}(\delta_2) \text{ for } \varepsilon_0 := \delta_2 \left(\max_{1 \leq i \leq r} |\boldsymbol{\beta}^i| \right)^{-1} > 0. \quad (56)$$

The fact that $\text{rank}(S^*(g)) = \text{rank}(I_f) = \text{rank}\{\boldsymbol{\nu}^1, \dots, \boldsymbol{\nu}^r\}$ then implies that for any $\boldsymbol{\nu} \in S^*(g)$ there exist $a_1(\boldsymbol{\nu}), \dots, a_r(\boldsymbol{\nu}) \in \mathbb{Q}$ such that $\boldsymbol{\nu} = a_1(\boldsymbol{\nu})\boldsymbol{\nu}^1 + \dots + a_r(\boldsymbol{\nu})\boldsymbol{\nu}^r$. It follows that for all $\mathbf{s} \in U_{-\delta_2}$, $G(f; \mathbf{s}) = W(\langle \boldsymbol{\nu}^1, \mathbf{s} \rangle, \dots, \langle \boldsymbol{\nu}^r, \mathbf{s} \rangle)$ where

$$W(\mathbf{z}) = \prod_p \left(\prod_{\boldsymbol{\nu} \in I_{\mathbf{c}}} \left(1 - \frac{1}{p^{1 + \sum_{i=1}^r a_i(\boldsymbol{\nu}) z_i}} \right)^{g(\boldsymbol{\nu})} \right) \left(1 + \sum_{\boldsymbol{\nu} \in S^*(g)} \frac{g(\boldsymbol{\nu})}{p^{\langle \boldsymbol{\nu}, \mathbf{c} \rangle + \sum_{i=1}^r a_i(\boldsymbol{\nu}) z_i}} \right). \quad (57)$$

So to conclude, it suffices to prove that W converges and defines a holomorphic function in $\mathcal{R}(\varepsilon_0)$. But for all $\mathbf{s}' \in \psi^{-1}(\mathcal{R}(\varepsilon_0))$, $\tilde{W}(\mathbf{s}') = W \circ \psi(\mathbf{s}') = W(\varphi(\mathbf{s}', \mathbf{0})) = G(f; (\mathbf{s}', \mathbf{0}))$, and

by definition of ε_0 we have $\psi^{-1}(\mathcal{R}(\varepsilon_0)) \times \{\mathbf{0}\}^{n-r} \subset \mathcal{R}(\delta_2) \times \{\mathbf{0}\}^{n-r} \subset U_{-\delta_2}$. It follows that $W := W \circ \psi$ converges and defines a holomorphic function in $\psi^{-1}(\mathcal{R}(\varepsilon_0))$. We deduce by composition that W converges and defines a holomorphic function in $\mathcal{R}(\varepsilon_0)$. This completes the proof of Proposition 2. \diamond

5.3 Proofs of Theorem 1

By symmetry we have: $Z_{H_P}(U(\mathbf{A}); s) := \sum_{M \in U(\mathbf{A})} H_P^{-s}(M) = c(\mathbf{A}) \sum_{\mathbf{m} \in \mathbb{N}^{n+1}} \frac{f(m_1, \dots, m_{n+1})}{P(m_1, \dots, m_{n+1})^{s/d}}$,

where $c(\mathbf{A})$ is the constant defined in (6), and f is the function defined by:

1. $f(m_1, \dots, m_{n+1}) = 1$ if $m_1^{a_{i,1}} \dots m_{n+1}^{a_{i,n+1}} = 1 \forall i = 1, \dots, l$ and $\gcd(m_1, \dots, m_{n+1}) = 1$
2. $f(m_1, \dots, m_{n+1}) = 0$ otherwise.

It is easy to see that f is a multiplicative function and that for any prime number p and any $\boldsymbol{\nu} \in \mathbb{N}_0^{n+1}$: $f(p^{\nu_1}, \dots, p^{\nu_{n+1}}) = g(\boldsymbol{\nu})$ where g is the characteristic function of the set $T(\mathbf{A})$ defined in §3.1. So, it is obvious that f is also uniform (see definition 3 in §3.2.3).

It now suffices to verify that the assumptions of Theorem 3 are satisfied.

By using the notations of §3.2.3, it is easy to check that

$$\mathcal{E}(f) = \mathcal{E}(T^*(\mathbf{A})) \text{ and } \mathcal{F}_0(f) = \mathcal{F}_0(\mathbf{A}).$$

Therefore, Proposition 2 implies that the first part of Theorem 1 follows from Theorem 3.

Let us now suppose that $\dim(\mathcal{F}_0(\mathbf{A})) = \dim X(\mathbf{A}) = n - l$. Since

$$\dim(\mathcal{F}_0(\mathbf{A})) = \text{rank}(\mathcal{F}_0(\mathbf{A}) \cap T^*(\mathbf{A})) - 1 \leq \text{rank}(T^*(\mathbf{A})) - 1 \leq \dim \mathbb{R}^{n+1} - \text{rank}(\mathbf{A}) - 1 = n - l,$$

it follows that $\dim(\mathcal{F}_0(\mathbf{A})) = \text{rank}(T^*(\mathbf{A})) - 1 = \text{rank}(S^*(g)) - 1$. Consequently, Proposition 2 implies that the second part of Theorem 1 also follows from Theorem 3, once one has also noted that Lemma 5 implies $\iota(f) = \iota(\mathcal{E}(f)) = |\mathbf{c}|$ for any normalized polar vector \mathbf{c} of $\mathcal{F}_0(f) = \mathcal{F}_0(\mathbf{A})$. \diamond

Proof of Theorem 2:

Let $A_n(\mathbf{a})$ be the $1 \times (n + 1)$ matrix $A_n(\mathbf{a}) := (a_1, \dots, a_n, -q)$. It is then clear that $X_n(\mathbf{a}) = V(A_n(\mathbf{a}))$. Thus, Theorem 2 will follow directly from Corollary 2 once we show that all the hypotheses of the corollary are satisfied.

Set $c(\mathbf{a}) := \frac{1}{2} \# \{(\varepsilon_1, \dots, \varepsilon_{n+1}) \in \{-1, +1\}^{n+1} \mid \varepsilon_1^{a_1} \dots \varepsilon_n^{a_n} = \varepsilon_{n+1}^q\}$.

As above, by symmetry we have for all $t > 1$:

$$N_{H_P}(U_n(\mathbf{a}); t) = \#\{M \in U_n(\mathbf{a}) \mid H_P(M) \leq t\} = c(\mathbf{a}) \sum_{\{\mathbf{m} \in \mathbb{N}^n; \tilde{P}(\mathbf{m})^{1/d} \leq t\}} f(m_1, \dots, m_n)$$

where \tilde{P} is the generalized polynomial defined by $\tilde{P}(X_1, \dots, X_n) := P(X_1, \dots, X_n, \prod_j X_j^{a_j/q})$

and f is the function defined by:

$$f(m_1, \dots, m_n) = 1 \text{ if } m_1^{a_1} \dots m_n^{a_n} \text{ is the } q^{\text{th}} \text{ power of an integer and } \gcd(m_1, \dots, m_n) = 1$$

and $f(m_1, \dots, m_n) = 0$ otherwise.

It is easy to see that f is a multiplicative function and that for any prime number p and any $\boldsymbol{\nu} \in \mathbb{N}_0^n$: $f(p^{\nu_1}, \dots, p^{\nu_n}) = g(\boldsymbol{\nu})$, where g is the characteristic function of the set $L_n(\mathbf{a}) \cup \{\mathbf{0}\}$ and $L_n(\mathbf{a}) := \{\mathbf{r} \in \mathbb{N}_0^n \setminus \{\mathbf{0}\}; q|\langle \mathbf{a}, \boldsymbol{\nu} \rangle \text{ and } \nu_1 \dots \nu_n = 0\}$. So it is clear that f is also uniform. By definition, we have that $\mathcal{E}(f) = \mathcal{E}(\mathbf{a}) = \mathcal{E}(L_n(\mathbf{a}))$. A simple check verifies that $\text{rank}(S^*(g)) = \text{rank}(L_n(\mathbf{a})) = n$. Moreover, the face $\mathcal{F}_0(\mathbf{a}) = \mathcal{F}_0(f)$ is compact. Consequently the assumption $\dim \mathcal{F}_0(f) = \text{rank}(S^*(g)) - 1$ is equivalent to the assumption that $\mathcal{F}_0(\mathbf{a})$ is a facet of $\mathcal{E}(\mathbf{a})$. As a result, Proposition 2 implies that Theorem 2 follows from Corollary 2, once one has also noted that Lemma 5 implies $\iota(f) = \iota(\mathcal{E}(f)) = |\mathbf{c}|$ for any normalized polar vector \mathbf{c} of $\mathcal{F}_0(\mathbf{a})$. \diamond

Acknowledgments: The author wishes to express his thanks to Ben Lichtin for his many helpful suggestions and his careful reading of this paper.

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